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A Comparison of the Jenq-Shah Two-Parameter Procedure and the Cohesive Zone Procedure for Predicting the Failure of Large Concrete Structures

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The Jenq-Shah two-parameter procedure, which is very easy to apply even with small structures, and the cohesive zone procedure, are procedures that can be used to predict the maximum load that a cracked concrete structure can sustain. The procedures are shown to be equivalent when the partially fractured zone is small compared with a concrete structure's characteristic dimension. ADVANCED CEMENT BASED MATERIALS 1995, 2, 85–90

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When a cracked concrete structure is loaded, a partially fractured zone develops ahead of the initial crack tip, and with a load control situation this zone increases in size until the load attains its maximum value when the structure fails in an unstable manner. In quantifying the instability condition and the maximum load, two main general methodologies have been employed: (1) cohesive zone procedures, and (2) effective, or equivalent elastic, crack procedures. Cohesive zone procedures, first applied to concrete by Hillerborg et al. [1], clearly recognize that there are unbroken ligaments of material within the partially fractured zone ahead of the initial crack tip. The ligament behavior is averaged so that the restraining effect of the ligaments is represented by a decreasing stress (p) versus increasing crack face opening (v) relationship. With the crack tip being defined as the leading edge of the partially fractured zone, there is therefore a cohesive, or softening, zone which is coplanar with the crack and which extends behind the actual crack tip to the initial crack tip position. It is assumed that the p-v law is a material characteristic, with the maximum value of p (i.e., p_c) occurring at the crack tip. The softening zone is said to be fully developed when the stress p falls to zero at the trailing edge of the cohesive zone (i.e., the initial crack tip position), assuming that this situation is attained when the relative displacement v attains a critical value δ_c. The maximum load is calculated on the basis of these assumptions, while recognizing that the load maximum is, in general, attained prior to the full development of the cohesive zone, and this complicates the analysis.

With effective crack models, the material behavior within the partially fractured zone is not represented by a stress-relative displacement relation. Instead, the partially fractured zone is characterized by an effective crack length, which is determined by imposing an additional instability condition. Prominent among the effective crack procedures is the two-parameter procedure proposed by Jenq and Shah [2], which is very easy to apply. With this procedure, the attainment of maximum load and system instability requires the simultaneous satisfaction of two conditions: (1) the linear elastic fracture mechanics (LEFM) stress intensity K_I at the tip of the effective elastic crack should equal a critical value K_*, and (2) the crack tip opening displacement at the initial crack tip should equal a critical value δ_*, which should not be confused with the parameter δ_c referred to in the preceding paragraph.

The objective of this article is to examine the relationship between the cohesive zone procedure and the two-parameter procedure, in the context of the behavior of a large structure where the partially fractured zone is small compared with the structure's characteristic dimensions. More particularly, the article compares the failure predictions obtained by these two procedures.

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Cohesive Zone Description

Consider a solid, deforming in accord with mode I plane strain conditions, in which there is a crack of initial depth a_o, D is a characteristic geometric dimension (say, for example, the solid width), B is the thickness, and σ_N is a nominal stress defined as P/BD, where P is the applied load. The linear elastic stress intensity factor K_I, defined with regard to a crack of depth a, can be expressed in the general form

$$K_I = \sigma_N \sqrt{D}\, S(a/D) \tag{1}$$

where $S(a/D)$ is a geometrical shape factor. Following Planas and Elices [3], the J integral, as defined by Rice [4], can be written in the form

$$J = \frac{1}{E_o}[K_I(a_o + \Delta a_E)]^2 \tag{2}$$

where $E_o = E/(1 - \nu^2)$, E being Young's modulus and ν being Poisson's ratio; Δa_E is the elastically equivalent size of cohesive zone, which need not necessarily be fully developed. Expanding eq 2 to the first two terms gives

$$J = \frac{K_{IN}^2}{E_o}\left(1 + \frac{2S_o'}{S_o}\cdot\frac{\Delta a_E}{D}\right) \tag{3}$$

where S_o is the value of the shape factor for $a = a_o$ and $S_o' = dS/da$ for $a = a_o$; K_{IN} is the stress intensity at the initial crack tip, defined by eq 1 but with $a = a_o$.

Now J, as given by eq 3, can be equated with the area under the P-v curve up to a value of v that is equal to the displacement δ at the trailing edge of the cohesive zone (i.e., the initial crack tip). Thus

$$\frac{K_{IN}^2}{E_o}\left(1 + \frac{2S_o'}{S_o}\cdot\frac{\Delta a_E}{D}\right) = W_F = \int_o^\delta p(v)dv. \tag{4}$$

Focusing attention on the behavior of positive geometries (i.e., those geometries for which the shape factor $S[a/D]$ increases with crack extension), Planas and Elices [3] have used general arguments to show that, for the case where $\Delta a_E/D$ is small, the maximum load is given via eq 4, but with $\Delta a_E \equiv R_E$ and $W_F \equiv G_F$. R_E is the elastically equivalent size of cohesive zone associated with a semiinfinite crack in a remotely loaded infinite solid, and G_F is the specific fracture energy of the material (i.e., $G_F = W_F$ as given by eq 4 with $\delta = \delta_c$). Consequently, the resulting expression giving the maximum load is, with $K_{IN,MAX}$ being the value of the stress intensity factor K_{IN} at maximum load,

$$\frac{E_o G_F}{K_{IN,MAX}^2} = 1 + \frac{2S_o'}{S_o}\cdot\frac{R_E}{D} \tag{5}$$

this expression being valid for small R_E/D. It should be emphasized that R_E in eq 5 is the elastically equivalent size of cohesive zone associated with a semiinfinite crack in a remotely loaded infinite solid ($D = \infty$), even though eq 5 refers to the case where R_E/D, though small, is not zero, as it is for the case where $D = \infty$. Bazant and Kazemi [5] have argued that this expression can be used for all R_E/D as an approximate representation of the true state of affairs.

Jenq-Shah Two-Parameter Description

It was indicated above that, with the Jenq-Shah two-parameter description, the attainment of maximum load requires the simultaneous satisfaction of two conditions: (1) the LEFM stress intensity K_I at the tip of the effective elastic crack should equal a critical value K_*, and (2) the crack tip opening displacement at the initial crack tip should equal a critical value δ_*. A great merit of the two-parameter procedure is that it is very easy to apply.

Now, with the preceding section's terminology, let K_E be the stress intensity at the tip of the effective elastic crack and Δa_{TP} be the distance between the initial crack tip and the effective crack tip. (Note that Δa_{TP} is not necessarily the same as the parameter Δa_E used in the preceding section). Consequently, the two conditions (1) and (2) are

$$K_E = K_* \tag{6}$$

and

$$\frac{4\, K_E}{\pi E_o}\sqrt{2\pi \Delta a_{TP}} = \delta_* \tag{7}$$

for any geometrical configuration where Δa_{TP} is small in comparison with other solid dimensions, although it should be emphasized that the two-parameter description is intended by the originators of the description to be applicable even when Δa_{TP} is not necessarily small. Elimination of K_E between eqs 6 and 7 gives

$$\Delta a_{TP} = \frac{\pi E_o^2\, \delta_*^2}{32\, K_*^2}\,. \tag{8}$$

Furthermore, with $K_E = K_I(a_o + \Delta a_{TP})$, expansion of eq 6 to the first two terms, and the use of eq 1, gives

$$K_{IN}\left(1 + \frac{S'_o}{S_o} \cdot \frac{\Delta a_{TP}}{D}\right) = K_* \tag{9}$$

where K_{IN} is the stress intensity defined with regard to the initial crack tip. Remembering that we are concerned with the case where Δa_{TP} is small, it follows that, when expressed in a format similar to eq 5 in the preceding section, the expression giving the maximum load is

$$\frac{E_o G_F}{K^2_{IN,MAX}} = \left(1 + \frac{2S'_o}{S_o} \cdot \frac{\Delta a_{TP}}{D}\right)\frac{E_o G_F}{K^2_*} \tag{10}$$

this expression being valid for small $\Delta a_{ATP}/D$, and noting that Δa_{TP} is given by eq 8.

Comparison of the Cohesive Zone and Two-Parameter Descriptions

The maximum load expressions derived via the cohesive zone and two-parameter descriptions are given by eqs 5 and 10, respectively. For these expressions to be equivalent, the following two conditions should both be satisfied:

$$K^2_* = E_o G_F \tag{11}$$

and

$$\Delta a_{TP} \equiv \frac{\pi E_o^2 \delta_*^2}{32 K_*^2} = R_E \tag{12}$$

remembering that R_E is the elastically equivalent size of cohesive zone associated with a semiinfinite crack in a remotely loaded infinite solid, and G_F is the specific fracture energy of the material for the relevant p-v law describing the behavior within a cohesive zone.

The two-parameter procedure claims that both K_* and δ_* are material parameters, that is, they are independent of geometry. Consequently, if there is to be consistency with the cohesive zone procedure, they should match with the corresponding parameters for a cohesive zone that is associated with a semiinfinite crack in a remotely loaded infinite solid, when the maximum load is associated with the attainment of a fully developed zone. That being the case, K_*^2 should be equal to E_o G_F and eq 11 is immediately satisfied. Furthermore, δ_* should be equated with δ_c and then eq 12 becomes, after substitution for $K_*^2 = E_o G_F$

$$R_E = \frac{\pi E_o \delta_c^2}{32 G_F} \tag{13}$$

and this condition must be satisfied in order to have equivalence between the two procedures. In other words, R_E should be given by eq 13 irrespective of the p-v law within the cohesive zone. The extent to which R_E can be represented by eq 13 is examined below.

Relation Between R_E and the Behavior Within a Cohesive Zone

R_E is the elastically equivalent size of fully developed cohesive zone associated with a semiinfinite crack in a remotely loaded infinite solid; it should be distinguished from R_A, the actual size of fully developed zone for the same situation. Planas and Elices [6] have derived a simple lower bound expression for R_E; it is, in fact, given by the right-hand side of eq 13, and in the ensuing considerations we examine the extent to which R_E conforms to this lower bound value for different behavioral characteristics within the cohesive zone.

Figure 1 shows the cohesive zone associated with a semiinfinite crack in a remotely loaded infinite solid, with the coordinate system being as indicated; it is assumed that the zone is fully developed in the sense that the relative displacement v of the crack faces at the trailing edge of the zone is such that there is no cohesion between the crack faces (i.e., $v = \delta_c$ when $p = 0$). The governing relationship between the stress p within the softening zone and the displacement v is [7]

$$\frac{dv}{ds} = \frac{4s^{1/2}}{\pi E_o}\int_0^{R_A} \frac{p(v)dt}{(s-t)t^{1/2}} \,. \tag{14}$$

Since the stress p at the crack tip ($s = 0$) has a finite value p_c, the stress intensity at the tip is zero, and

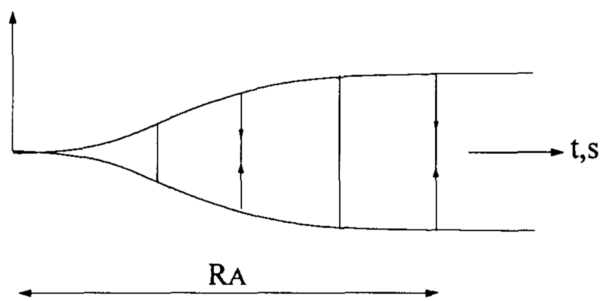

FIGURE 1. The cohesive zone associated with a semiinfinite crack in a remotely loaded infinite elastic solid. R_A is the size of the fully developed cohesive zone.

consequently the stress intensity K_∞ due to the remotely applied loadings must equate with the stress intensity due to the cohesive stresses, that is,

$$K_\infty = \frac{2^{1/2}}{\pi^{1/2}} \int_0^{R_A} \frac{p(v)ds}{s^{1/2}} . \tag{15}$$

Inversion of the integral eq 14, coupled with the use of eq 15, leads to the expression

$$K_\infty = \left(E_o \int_0^{\delta_c} p(v)dv\right)^{1/2} = (E_oG_F)^{1/2} \tag{16}$$

where G_F is the specific fracture energy of the material for the p-v law in question; this relation can also be obtained directly by using Rice's J path independent integral procedure [4].

Following procedures that have been described in greater detail elsewhere [8], instead of working directly with a prescribed p-v law, it will be assumed that there is a power law relationship between the stress p and distance s as measured along the cohesive zone, with $p = p_c$ at the crack tip and $p = 0$ at the trailing edge of the fully developed cohesive zone. Equation 14 then leads to a relation between the relative displacement v and distance s, which, when coupled with the assumed p-s relation, gives a p-v law for which R_A can be determined by imposing the boundary condition that $v = \delta_c$ at the trailing edge of the cohesive zone. R_E is then obtained from the general $R_E - R_A$ relation that has been derived by Planas and Elices [3]:

$$\frac{R_E}{R_A} = \frac{\int_0^1 (1-u)k(u)du}{\int_0^1 k(u)du} \tag{17}$$

where, for the case where p tends to zero continuously as $u \to 1$, the function $k(u)$ is related to the stress within the cohesive zone via the relationship

$$k(u) = \frac{1}{\pi} \int_u^1 (w-u)^{-1/2} \frac{dp}{dw} dw \tag{18}$$

with $u = s/R_A$ and $w = t/R_A$; s and t are the distance as measured from the leading edge of the cohesive zone (see Figure 1).

We will assume that the stress (p) – distance (u,s) relation within the cohesive zone has the form

$$\frac{p}{p_c} = (1-u)^{n+1/2} \tag{19}$$

with n having any value greater than $-½$. This relation is such that the stress is equal to p_c at the leading edge of the cohesive zone ($u = 0$), is positive throughout the cohesive zone, and is zero at the trailing edge of the cohesive zone ($u = 1$), which is assumed to be fully developed. Substitution of eq 19 in eq 14 then gives

$$\frac{dv}{du} = \frac{4p_cR_A}{\pi E_o} u^{1/2} J_n (1-u) \tag{20}$$

with $J_n(u)$ being given by the integral relation:

$$J_n(u) = \int_0^1 \frac{w^{n+1/2}\, dw}{(w-u)(1-w)^{1/2}} . \tag{21}$$

Equations 20 and 21, when coupled with the boundary condition $v/\delta_c = 1$ when $u = 1$, give the result

$$\frac{p_cR_A}{E_o\delta_c} = 2^{2n-1} \frac{[\Gamma(n+2)]^2 (2n+3)}{[\Gamma(2n+3)]\,(2n+2)} \tag{22}$$

while eqs 15, 16, and 19 give the result

$$\frac{G_F}{p_c\delta_c} = \frac{\pi[\Gamma(2n+3)](2n+3)}{2^{2n+4}[\Gamma(n+2)]^2 (2n+2)} . \tag{23}$$

With regard to the determination of R_E, it is first necessary to determine the function $k(u)$, which is given by eq 18. Thus, substitution of eq 19 into eq 18 gives

$$k(u) = -\frac{(n+½)\,p_c}{\pi} \int_u^1 (w-u)^{-1/2} (1-w)^{n-1/2} dw \tag{24}$$

whereupon eqs 17 and 24 give, after manipulation of the integrals, the very simple expression

$$\frac{R_E}{R_A} = \frac{n+1}{n+2} \tag{25}$$

a result which is valid for all $n > -½$. It immediately follows from eqs 22 and 25 that

$$\frac{p_cR_E}{E_o\delta_c} = 2^{2n-1} \frac{[\Gamma(n+2)]^2 (2n+3)}{[\Gamma(2n+3)]\,(2n+4)} \tag{26}$$

a result which is again valid for all $n > -\frac{1}{2}$. Finally, eqs 23 and 26 give the result

$$R_E = \frac{\pi E_o \delta_c^2}{32G_F} \cdot \frac{1}{\left(1 - \frac{1}{(2n+3)^2}\right)} \tag{27}$$

an expression that is valid for all $n > -\frac{1}{2}$. Table 1 shows R_E for various n values, together with the value of G_F as given by eq 23. These results clearly show that the lower bound value, as determined by Planas and Elices [6], namely $\pi E_o\delta_c{}^2/32G_F$, provides a reasonably accurate estimate of R_E, particularly when the softening is pronounced within the cohesive zone (i.e., when $G_F/p_c\delta_c$ is small); thus, when $G_F/p_c\delta_c < 0.4$, the lower bound value is accurate to within ~5%.

To support this view, we present results that have been obtained [9] for the idealized piecewise p-v law shown in Figure 2, a law which, of course, gives a distribution of stress within the cohesive zone that is different to the case of a power function law considered earlier. When q and λ are both small, it has been shown that

$$R_E = \frac{\pi E_o \delta_c}{24 p_c} \cdot \frac{[q + 3(\lambda + \sqrt{q\lambda + \lambda^2})]}{[q + (\lambda + \sqrt{q\lambda + \lambda^2}][q + 2(\lambda + \sqrt{q\lambda + \lambda^2})]} \tag{28}$$

while

$$\frac{G_F}{p_c\delta_c} = (q + \lambda). \tag{29}$$

It therefore follows from eqs 28 and 29 that

$$R_E = \frac{\pi E_o \delta_c^2}{32G_F} \cdot \frac{4}{3m^2}\,[(m+1)(m-2) + 2\sqrt{m+1}] \tag{30}$$

with $m = q/\lambda$. Table 2 shows R_E for various m values, and the results clearly show that the lower bound value provides a reasonably accurate estimate of R_E, particularly when most of the specific fracture energy is associated with the initial region of the p-v law.

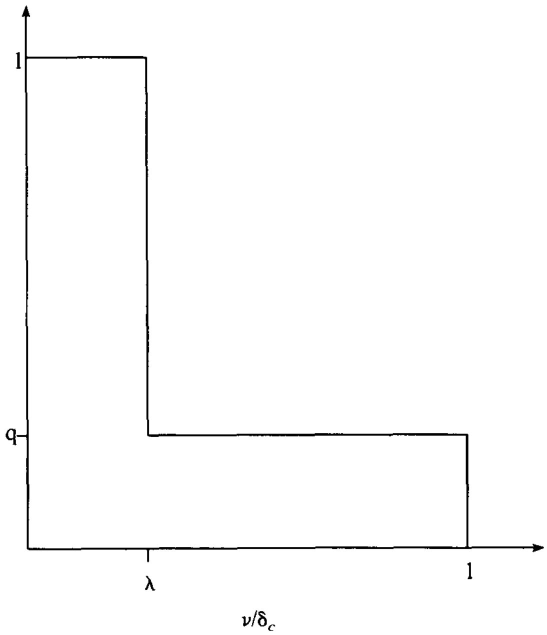

FIGURE 2. A simple piecewise p-v law.

To put the results (Tables 1 and 2) in perspective, it should be observed that the situation where there is an initially steep decline in the p-v law followed by a long tail, as is often assumed to be the case when modeling the behavior of concrete, can be simulated by assuming that n is large with the power law stress-distance variation (results in Table 1), or when q and λ are both small (results in Table 2).

Concluding Comments

For the case where the partially fractured zone is small in comparison with the characteristic dimension of a concrete structure, this article has shown that the Jenq-Shah two-parameter procedure is essentially equivalent, for practical purposes, to the cohesive zone procedure, as regards the prediction of the maximum load that a cracked structure can sustain in a load control

TABLE 1. The parameter R_E for various n values, together with the corresponding G_F value, for the power law stress-distance variation

n	$-\frac{1}{2}$	0	1	2	3	4	5
$\frac{G_F}{p_c\delta_c}$	1.000	0.588	0.367	0.285	0.241	0.212	0.191
$\frac{R_E}{\left[\frac{\pi E_o\delta_c^2}{32G_F}\right]}$	1.333	1.125	1.042	1.021	1.012	1.008	1.006

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TABLE 2. The parameter R_E for various m ($=q/\lambda$) for values for the piecewise p-v stress-displacement law; q and λ are both small

m	∞	5	2	1	0.5	0.2	0
$\dfrac{R_E}{\left[\dfrac{\pi E_0 \delta_c^2}{32 G_F}\right]}$	1.333	1.221	1.154	1.104	1.064	1.029	1.000

situation. The Jenq-Shah procedure, a great merit of which is that it is easy to apply, requires the simultaneous satisfaction of two conditions: (1) the LEFM stress intensity at the tip of the effective elastic crack should equal a critical value K_*, and (2) the crack tip opening displacement at the initial crack tip should equal a critical value δ_*. K_* and δ_* are presumed to be material parameters, that is, they are independent of a structure's geometrical dimensions. On the other hand, the cohesive zone procedure presumes that the relationship between the stress p and relative displacement v within the cohesive zone is a material characteristic, that is, it is independent of geometrical parameters. However, the article has shown that when the partially fractured zone size is small compared with a solid's characteristic dimension, the two procedures are essentially equivalent for practical purposes. Otherwise, in general, the agreement between the procedures' predictions will be approximate, although the degree of approximation can be improved by judicious selection of the input parameters.

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