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Prediction of Stresses in Concrete Pavements Subjected to Non-linear Gradients

Ashraf R. Mohamed & Will Hansen

Department of Civil Engineering, 2430 GG Brown Building, Ann Arbor, MI 48109-2125, USA

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Abstract

This paper presents a technique for analyzing the residual stresses in concrete pavements subjected to non-linear (stress or strain) gradients throughout the slab thickness. The analysis is separated into two parts. In the first part, an expression is presented for calculating the self equilibrated stresses within a cross-section due to internal restraints (i.e. satisfying equilibrium conditions and continuity of the strain field within the cross-section). These stresses are independent of slab dimensions and boundary conditions. In step two, the stresses due to external restraints (i.e. self-weight and subgrade reaction) are calculated using an equivalent linear temperature gradient obtained from the first part and using existing closed form solutions by Westergaard [Westergaard, H. M., Computation of stresses in concrete roads. In Proc. of the 5th Annual Meeting, *Vol. 5, Part I, Highway Research Board, 1926, pp. 90–112] or Bradbury [Bradbury, R. D.,* Reinforced Concrete Pavements. *Wire Reinforcement Institute, Washington D.C., 1947.]. The solution to this step includes slab length and boundary conditions. Total internal stresses due to non-linear gradients are obtained using the superposition principle. The proposed method has been applied to field data from another study for varying temperature profiles within a 24 h period and compared to results from conventional analysis assuming linear gradients. Significant differences were found between the two methods for night-time and early-morning conditions. A linear gradient solution sometimes underestimates tension in the bottom of a slab prior to vehicle loading by a factor of three. © 1997 Elsevier Science Limited*

INTRODUCTION

It is well-known that the stresses affecting concrete pavements are the result of the combined effects from repeated wheel loading and varying temperature and moisture gradients. Figure 1 illustrates schematically the location of tensile stresses and curvature for temperature, moisture and wheel loading conditions.

When the temperature across the slab thickness exists as a gradient, the slab's response will

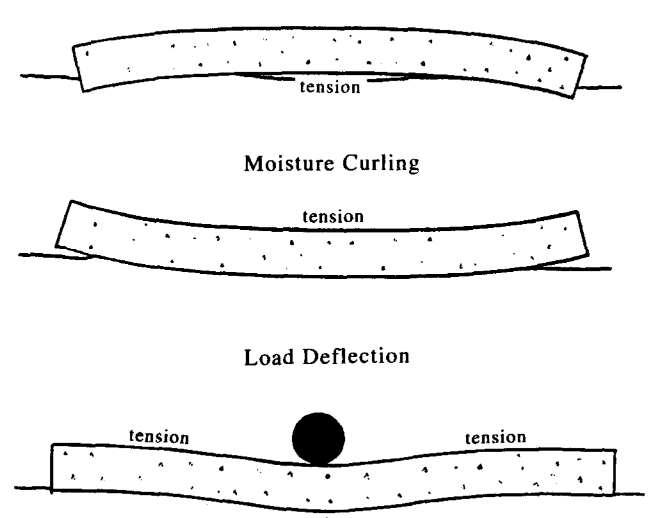

Fig. 1. Location of tensile stress under temperature-, moisture- and load-related curvatures.

be manifested as either an upward or downward movement of the slab edges relative to the center depending on the time of day, as illustrated in Fig. 1. As the top surface of the slab becomes cooler than its bottom, the slab tends to curve with its edges upward. Downward curvature may occur during night-time conditions. The slab deformation due to temperature gradients is known as curling. Similarly warping is the slab deformation from moisture gradients. Since the slab is not free to deform without internal and external restraints, internal stresses will develop within the slab cross-section.

The AASHO Road Test[1] established that repeated curling and warping stresses can be significant depending on the pavement geometry and stiffness of the subgrade, and can result in cracking. Slab length was found to be the major factor in controlling cracking from curling and warping.

The problem of a concrete slab on grade subjected to a *linear temperature gradient* has been solved by Westergaard[2,3] and Bradbury[4] for the cases of infinite, semi-infinite, infinite with constant width, and finite slabs. According to these models the temperature stresses alone may be as high as the stresses due to loading.

It is now well-established from field measurements that the actual distribution of both the temperature and shrinkage gradients through the thickness of a slab is non-linear.[5–8] Numerically-based solution methods for calculating the effect of *non-linear gradients* on internal stresses in concrete pavements have been developed.[9]

This study will discuss a new methodology of analyzing non-linear profiles from curling and warping for internal pavement stresses.

BASIC EQUATIONS

The general solution for an elastic slab, curved in two directions, subjected to *linear temperature* profile was given by Westergaard[2] based on plate theory as

$$-\frac{\partial^2 w}{\partial x^2} = \frac{12}{Eh^3}(Mx - \nu My) + \frac{\alpha \Delta t}{h} \quad (1)$$

$$-\frac{\partial^2 w}{\partial y^2} = \frac{12}{Eh^3}(My - \nu Mx) + \frac{\alpha \Delta t}{h} \quad (2)$$

The displacement w is in the z-direction (Fig. 2), h is the slab thickness, E, ν, and α are, respectively, Young's modulus, Poison's ratio, and the thermal expansion coefficient of the slab.

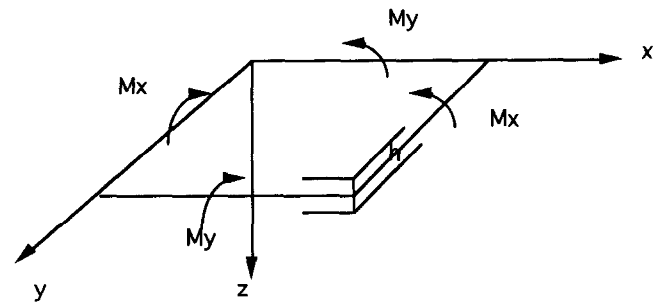

Fig. 2. Coordinate system and positive directions and forces.

The right-hand sides of the first and second equations represent the total curvature of the slab. The first part corresponds to moment curvature in both directions. The second part is the curvature due to the linear temperature gradient. In these equations, positive curvature is considered with the positive moment (i.e. when the slab is curved with its edges upward). This curvature is associated with negative temperature gradient (i.e. the temperature at the top surface is lower than at the bottom). It should be noted that these equations represent a totally *unrestrained* slab. If the slab is restrained externally, appropriate boundary conditions should be used to solve the differential equations.

NON-LINEAR PROFILES

The analysis of a non-linear profile (stress or strain) within a concrete slab is separated into two parts. The first part calculates the residual stresses due to the internal restraint. The second part estimates the stresses in the slab due to the external restraint (self weight and subgrade reaction). The first part of the solution will be derived in a closed form solution, while the second part requires the calculation of the equivalent linear temperature gradient and solving for stresses using either closed form solutions such as those by Westergaard or Bradbury, or available finite element packages such as ILLI-SLAB.

In deriving the general theory the following assumptions are made:

The slab is elastic, homogeneous, and isotropic with temperature independent material properties.

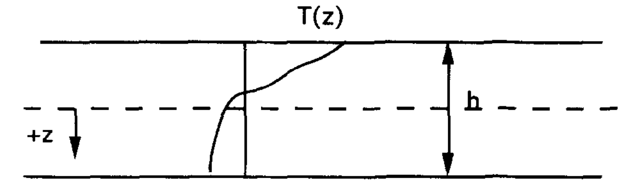

Fig. 3. Temperature distribution through thickness.

Plane sections remain plane after bending.

Stresses and strains in the z-direction are zero.

The deflection of the slab is small compared to the slab dimensions.

Temperature or shrinkage strains vary only across the cross-section (i.e. are independent of x and y).

Sign conventions used throughout are:

Positive curvatures and moments are those that cause the slab to bend upward at the edges.

Strain increase is considered positive.

Stresses are considered positive in tension and vice versa.

The coordinate system is such that z is considered positive downward and measured from the mid-plane as shown in Fig. 3.

Consider a fully unrestrained weightless slab subjected to a strain distribution $\varepsilon(z)$ across the slab thickness h as shown in Fig. 3. This strain profile could be due to a temperature gradient ($\varepsilon(z) = \alpha T(z)$) or a shrinkage gradient ($\varepsilon(z) = \varepsilon_{sh}(z)$). Since the slab is free to deform the strain profile due to such a gradient will be identical to the gradient profile, i.e.

$$\varepsilon_x = \varepsilon(z) \qquad (3)$$

$$\varepsilon_y = \varepsilon(z)$$

If the strain distribution is not uniform, different elements of the body will deform by an amount proportional to the local strain. This will conflict with the continuity requirements of the body. Thus, each element will exert a restraining action upon the movement of the surrounding elements, which will result in a continuous displacement field satisfying the continuity requirements. This system of strains will be accompanied by a system of self-equilibrated stresses known as the residual stresses.

General derivation for residual stresses

Assume that the slab is *virtually* restrained against deformation. Then the stress distribution required for such restraint is simply the same as the strain distribution in eqn (3) with opposite sign. The magnitude can be calculated by applying Hook's law and considering the stress state in two dimensions:

$$\sigma_x = \frac{-E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \qquad (4)$$

$$\sigma_y = \frac{-E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

Since the distribution of $\varepsilon(z)$ is independent of x and y, i.e., $\varepsilon_x = \varepsilon_y$, therefore:

$$\sigma = \sigma_x = \sigma_y = \frac{-E}{1-\nu^2}(1+\nu)\varepsilon(z) \qquad (5)$$

and

$$\sigma = \frac{-E}{1-\nu}\varepsilon(z) \qquad (6)$$

Equation (6) represents the stress distribution required to totally suppress the deformation of the slab.

Integrating eqn (6) over the slab thickness will result in the *virtual* normal force N required to prevent deformation, while integrating for moment will yield the corresponding *virtual* moment M about the mid-plane ($z = 0$). Thus

$$N \int_{-h/2}^{h/2} \sigma \, dz = \frac{-E}{1-\nu} \int_{-h/2}^{h/2} \varepsilon(z) \, dz \qquad (7)$$

or

$$N = \frac{-E}{1-\nu} N^* \qquad (8)$$

and

$$M = -\int_{-h/2}^{h/2} \sigma z \, dz = \frac{E}{1-\nu} \int_{-h/2}^{h/2} \varepsilon(z) z \, dz \qquad (9)$$

where the minus sign is introduced to take care of the sign conventions mentioned before (z is positive downward), thus

$$M = \frac{E}{1-\nu} M^* \qquad (10)$$

where N^* and M^* are constants resulting from the following integration:

$$N^* = \int_{-h/2}^{h/2} \varepsilon(z)\, dz \quad (11)$$

and

$$M^* = \int_{-h/2}^{h/2} \varepsilon(z) z\, dz \quad (12)$$

Equilibrium requirements

After obtaining the normal force and moment, virtually restraining the deformation of the slab, equilibrium of the internal forces should be maintained. Since the previously calculated resultants are not in equilibrium, imposing opposite and equal resultants N and M will ensure the equilibrium of the slab. Thus, the residual stresses can be calculated by adding the developed stresses due to such equilibrium resultants to those required for suppression of the deformation (i.e. thermal or shrinkage), thus

$$\sigma_{\text{res}} = \sigma + \frac{12(-M)}{h^3}(-z) + \frac{(-N)}{h} \quad (13)$$

where the negative signs with M and N are for opposite direction and with z for sign convention. Then, using eqns (11) and (12), the final form of the residual stress equation reads

$$\sigma_{\text{res}} = \frac{E}{(1-\nu)}\left[-\varepsilon(z) + \frac{12M^*}{h^3}(-z) + \frac{N^*}{h}\right] \quad (14)$$

It can be shown that eqn (14) satisfies the equilibrium requirements by integrating the residual stresses across the slab cross-section.

It should be noted that the residual stresses are produced internally to maintain equilibrium. Those stresses exist regardless of the external restraints of the slab and its boundary conditions. In other words, if the slab is left in the air without any external restraints while subjected to non-linear gradient, the slab will suffer only such residual stresses due to internal restraint. Furthermore, the slab will curve due to the equilibrium moment. However, if the slab is also restrained externally against such curvature by means of self weight and subgrade reaction, stresses will develop and should be added to the residual stresses to obtain the complete solution. The slab curvature will be calculated in the next section from which the equivalent linear temperature gradient can be obtained.

STRESSES DUE TO EXTERNAL RESTRAINT

Equivalent curvature calculation

For a slab bent in two directions, the relation between the moment M and curvature κ is

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = D \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} \quad (15)$$

where the plate rigidity $D = Eh^3/[12(1\text{-}\nu^2)]$. If we consider the case of strain changes in the z-direction only, which leads to

$$M_x = M_y = M \text{ and } \kappa_x = \kappa_y = \kappa$$

hence eqn (15) may read

$$M = D(1+\nu)\kappa \quad (16)$$

or

$$\kappa = \frac{M}{D(1+\nu)} \quad (17)$$

Substituting eqn (10) into eqn (16) can get the general form of the curvature as

$$\kappa = \frac{12M^*}{h^3} \quad (18)$$

The equivalent linear temperature gradient

The main concept in calculating the equivalent temperature gradient is to obtain the temperature difference between the bottom and top of the slab which produces the same curvature (see Westergaard solution, e.g. eqns (1) and (2)). This temperature difference could be obtained as

$$\Delta T_{\text{eq.}} = -\frac{h\kappa}{\alpha} \quad (19)$$

or

$$\Delta T_{\text{eq.}} = -\frac{12M^*}{\alpha h^2} \quad (20)$$

where the minus sign is introduced to account for the different sign convention of ΔT and κ as stated before. The importance of eqn (20) lies in the fact that the equivalent linear temperature gradient, which is the input of any closed form or numerical analysis, can be calculated directly based on the determination of the constant M^*. This simplifies the calculation of stresses due to non-linear temperature or shrinkage profiles into a problem of linear temperature variation which is well-understood.

STEP BY STEP CALCULATIONS

The general procedure to solve for the non-linear temperature profiles can be summarized as follows:

Obtain the strain distribution ($\alpha T(z)$ or $\varepsilon_{sh}(z)$) as a function of z being in the mid-plane (by curve-fitting the actual data using polynomial function).

Calculate the N^* and M^* according to eqns (11) and (12) (or integrate the actual data numerically as in Ref. 6).

Calculate the residual stress distribution according to eqn (14).

Calculate the equivalent linear temperature gradient $\Delta T_{eq.}$ according to eqn (20).

Calculate the stresses due to $\Delta T_{eq.}$ using closed form solution or numerically (i.e. finite element programs).

After solving the linear problem, add the solution of step 3 to that of step 5 to get the final stresses.

To illustrate the solution methodology, two examples will be considered:

Example 1

In order to validate the previously derived equations, the linear temperature gradient will be examined since the solution for the curvature and residual stresses are known in advance.

Consider the linear temperature gradient shown in Fig. 4, the temperature distribution as a function of z will be

$$T(z) = T_0(2z/h)$$

and hence

$$\varepsilon(z) = \alpha T_0(2z/h)$$

and the constants calculated based on eqns (11) and (12) are

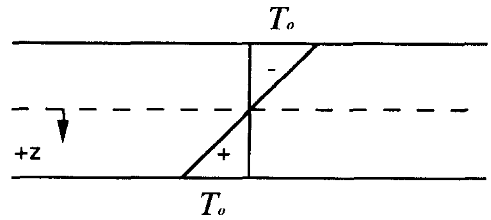

Fig. 4. Linear temperature gradient.

$$N^* = \int_{-h/2}^{h/2} -\alpha T_0(2z/h)\,dz = 0$$

$$M^* = \int_{-h/2}^{h/2} \alpha T_0(2z/h)z\,dz = \alpha T_0 h^2/6$$

Thus the residual stresses are

$$\sigma_{res} = \frac{E}{(1-\nu)}\left[-\alpha T_0(2z/h) + \frac{12\alpha T_0 h^2/6}{h^3}(z) + \frac{0}{h}\right] = 0$$

and the curvature becomes

$$\kappa = \frac{2\alpha T_0}{h}$$

which agrees with the known answer.

Example 2: application to field data

First the method developed by Bradbury for handling a linear temperature profile will be demonstrated in this example and applied to actual data documented by Thompson *et al.* for field measurements of the temperature distribution in a 9 in. PCC slab during April in Urbana, Illinois.[5] An attempt is made to fit such data with the best curve. Although it was mentioned by Richardson *et al.*[6] that the best fitting curve for temperature distribution is a second degree parabola, it is found to be polynomial of third degree of general form

$$T(z) = A + Bz + Cz^2 + Dz^3$$

Where the constants A, B, C and D are given in Table 1 for different times for the given data, while the simulated curves and actual data are shown in Fig. 5. On integrating the previous equation for N^* and M^*, one obtains, assuming $\alpha = 5 \times 10^{-6}$/°F, $E = 4 \times 10^6$ psi, $\nu = 0{\cdot}15$ and

Table 1. Comparison between linear and non-linear solutions

Time		2:00 a.m.	6:00 a.m.	10:00 a.m.	3:00 p.m.	7:00 p.m.	11:00 p.m.
Curve fitting constants	A	2·6905	1·881	1·2143	1·4762	2·9286	3·3214
	B	−0·4947	0·09127	0·171967	−0·5291	−1·3717	−0·87235
	C	−0·16402	−0·16931	0·079365	0·26852	0·079365	−0·11905
	D	0·032922	0·012346	−0·041152	−0·05144	0·020576	0·03479
Nonlinear temperature analysis							
Equiv. curvature	1/in	$-4·735\times10^{-7}$	$1·206\times10^{-6}$	$-1·64\times10^{-6}$	$-5·77\times10^{-6}$	$-5·612\times10^{-6}$	$-2·25\times10^{-6}$
Equiv. linear temp.	°F	0·852	−2·1714	2·952	10·4	10·1	4·05
Curvature stresses (psi)	Top	−10·5	26·7	−36·24	−128	−124	−49·7
	Bottom	10·5	−26·7	36·24	128	124	49·7
Residual stresses (psi)	Top	80·338	64·37	−60·5	−129·5	−7·56	67·66
	Bottom	23·866	43·193	10·08	−41·11	−42·9	7·98
Total stresses (top)	psi	69·838	91·07	−96·75	−257	−131·56	17·96
Total stresses (bottom)	psi	34·366	16·5	46·32	86·9	81·1	57·68
Stresses calculated based on linear temperature gradient							
Actual temp. diff.	°F	−1·5	−3	6	14	8·5	1·5
Curvature	1/in	$8·33\times10^{-7}$	0·00000166	$-3·33\times10^{-6}$	$-7·77\times10^{-6}$	$-4·72\times10^{-6}$	$-8·33\times10^{-7}$
Max top stresses	psi	18·414	36·83	−73·66	−172	−105	−18·4
Max bottom stresses	psi	−18·414	−36·83	73·66	172	105	18·4

1°F = (F-32)/1·8°C; 1 psi = 6·89 kPa; 1 in. = 2·45 cm.

coefficient of subgrade reaction $\kappa = 100$ psi/in., for the 9 in. thickness

$$N^* = 9A + 60·75C \text{ and } M^* = 60·75B + 738·1125D$$

This in turn leads to the residual stress equation as

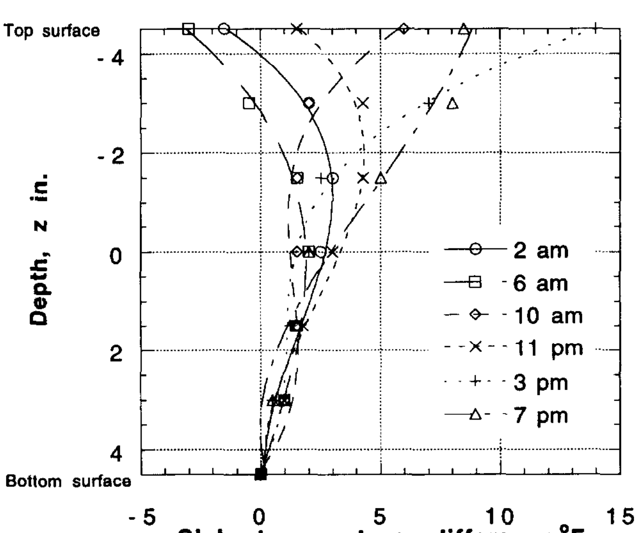

Fig. 5. Modeling of the actual data using third degree polynomial.

$$\sigma_{res} = 23·53[-Cz^2 - Dz^3 = +12·15Dz + 6·75C]$$

and curvature of

$$\kappa = \frac{12\alpha M^*}{h^3} = 8·23\times10^{-8}$$

$$[60·75B + 738·1125D]$$

and an equivalent temperature gradient of

$$\Delta T_{eq.} = -\frac{12M^*}{\alpha h^2} = -0·14814$$

$$[60·75B + 738·1125D]$$

To calculate the total stresses, consider a PCC slab 24 ft wide (two 12 ft lanes) × 30 ft in length. The stresses due to the equivalent linear temperature profile will be calculated using the Bradbury solution. The parameters are calculated as: $l = 39·71$ in. with $Lx = 24$ ft and $Ly = 30$ ft, $Cx = 1·0$ and $Cy = 1·05$. Only the interior stresses in the y-direction are calculated (since they are the maximum). The stress values at the top and bottom of the slab are given in Table 1 under the non-linear temperature section. For comparison, the stresses are calculated

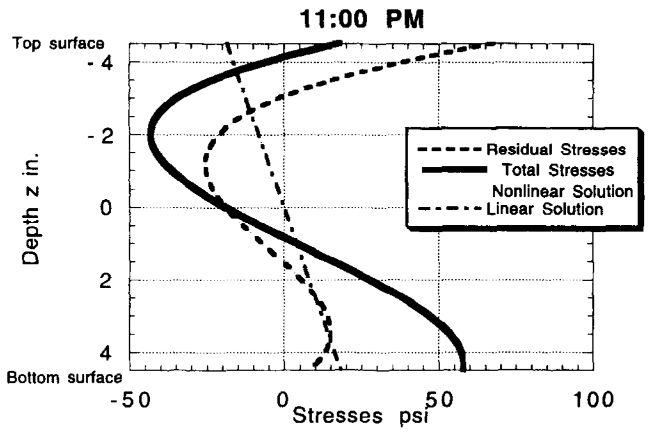

Fig. 6. Stress distribution through thickness at 11·00 p.m.

using the traditional method by calculating the stresses due to the difference in temperature between the top and bottom, and assuming linear gradient. This solution is also given in the same table under linear solution.

Comparison between non-linear temperature gradient solution and linear solution

The residual and total stress distributions are shown in Fig. 6 together with the linear solution for the case which results in the most discrepancy between the two methods (i.e. night time condition). Non-linear analysis predicts tension both at the slab top and bottom, whereas linear analysis predicts tension at the bottom only. The difference in the tensile stress predictions is about three times. More work is needed to compare results between the two methods using field measurements of temperature gradients.

CONCLUSIONS

A two-step procedure is presented for calculating the stress distribution in a concrete slab due to non-linear gradients from curling and warping. In the first step, the residual stresses of internal restraints are calculated. These stresses are independent of slab dimensions and boundary conditions. In the second step, the stresses due to external restraints are calculated using an equivalent linear temperature gradient obtained from the first step and using an existing closed form solution (i.e. Westergaard or Bradbury) or numerically (i.e. finite element programs). In step two, slab dimensions and boundary conditions are needed. Total stresses are obtained using the superposition principle.

The method was applied to measured temperature profiles by Thompson *et al.* during a 24 h period. The thermal stress predictions for night-time conditions especially show significant differences between linear and non-linear stress profiles. For this particular case, using linear stress profiles, the bottom tensile stresses are underestimated by a factor of three.

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