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Plasticity Approach to Shear Design

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Abstract

The paper presents some plastic models for shear design of reinforced concrete beams. Distinction is made between two shear failure modes, namely web crushing and crack sliding. The first mentioned mode is met in beams with large shear reinforcement degrees. The mode of crack sliding is met in non-shear reinforced beams as well as in lightly shear reinforced beams. For such beams the shear strength is determined by the recently developed crack sliding model. This model is based upon the hypothesis that cracks can be transformed into yield lines, which have lower sliding resistance than yield lines formed in uncracked concrete. Good agreement between theory and tests has been found. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords: design, plasticity, reinforced concrete, reinforcement, shear, web crushing

NOMENCLATURE

a	shear span/half of the span length
a/h	shear span ratio
A_s	longitudinal reinforcement area
A_{sw}	total cross-sectional area of one stirrup in a horizontal section
b	web width of beam
c	cohesion of concrete
c'	cohesion in a crack
f_c	uniaxial compressive strength of concrete
f_{tef}	Effective plastic tensile strength of concrete
f_{yw}	yield strength of shear reinforcement
h	depth of beam
h_i	lever arm, i.e. the distance between the tensile and the compression resultant
P	external load
P_{cr}	cracking load
P_u	ultimate load/load-carrying capacity
s	stirrup distance
$s(h)$	size effect parameter
u	relative displacement in yield line
V	shear force
W_I	internal work at failure
W_l	internal work at failure per unit length
x	Horizontal projection of yield line/critical diagonal crack

Greek letters

α	angle between yield line and displacement direction
φ	angle of friction
v	effectiveness factor for web crushing
v_0	effectiveness factor for non-shear reinforced beams and lightly reinforced beams
v_s	effectiveness factor for crack sliding
θ	inclination of the diagonal compression field
ρ	reinforcement ratio ($=A_s/bh$)
ρ_w	shear reinforcement ratio ($=A_{sw}/bs$)
σ_c	concrete stress
τ	shear stress
τ_c	$=0.059 v_0 f_c$
τ_u	shear capacity/shear strength of cracked concrete
ψ	shear reinforcement degree ($=\rho_w f_{yw}/f_c$)
ψ_{min}	minimum shear reinforcement degree

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INTRODUCTION

The application of plastic theory to shear in reinforced concrete has now been studied for more than 30 years. All important problems have been touched upon.

For many years the application of plastic theory in practical design was restricted to Denmark and Switzerland. Now the interest in other countries is growing and plastic methods have been adopted in the Eurocode 2, the future common concrete code in Europe.

The problem to be overcome is not to find the plastic solutions, although this is not always easy. The difficult problem is the pronounced softening of unconfined concrete which renders the perfectly plastic material model a rather crude one. What has been shown in the past decades is that by introducing empirically determined reduction factors, the so-called effectiveness factors, on the concrete strengths the theoretical solutions may be brought into close agreement with experiments.

We might say that the plastic theory is an extremely effective way of finding empirical formulas for the strength of a concrete structure subjected to shear.

The present ideas about the reduced strength of concrete in a concrete structure are described in the paper. Further, some important solutions from a practical point of view will be given. They will mainly be related to beam problems.

THE WEB CRUSHING CRITERION

The basic idea of plastic theory for reinforced concrete beams was formulated in 1967.[1] During the 1970s, further development of the theory took place resulting in, among other things, the establishment of the upper bound technique.

A detailed description of the theory may be found, for example, in Refs 2 and 3. Readers with interest in the historical aspects of the developments are referred to Ref. 4. In this paper, only a brief summary will be given of the earlier work.

In the plastic approach, concrete is assumed to be a rigid, perfectly plastic material obeying the modified Coulomb failure criterion with the associated flow rule. The tensile strength is normally put to zero and the effective compressive strength is νf_c, where ν is an effectiveness factor and f_c is the standard uniaxial cylinder strength.

The reinforcement material is assumed to be rigid, perfectly plastic as well with the yield strength f_y. Furthermore, the stirrups are normally assumed to be closely spaced so that their action may be described by an equivalent stirrup stress.

The plastic approach for design purposes was developed as a lower bound approach, i.e. it operates with statically admissible and safe stress fields. However, in some cases, coinciding upper and lower bound solutions have been found, see Ref. 3.

The lower bound solution for constant shear force

Consider a simply supported beam symmetrically loaded by two concentrated forces as shown in Fig. 1. The compression zone and the tensile zone are idealized as stringers carrying the forces C and T respectively. The shear force V is carried by the web with the thickness b. The depth of the web is the distance between the stringers. In design situations, as well as when comparing with test results, the depth has been taken as the internal moment arm h_i at maximum moment.

The shear force V gives rise to an average shear stress τ in the cross-section given by

$$\tau = \frac{V}{bh_i} \tag{1}$$

A statically admissible stress field in the web region is easily constructed. The concrete is assumed to be in a state of plane stress and subjected to uniaxial compression with the compression stress σ_c at an angle θ with the horizontal x-axis (the diagonal compression field). The horizontal components of this stress field are transferred to the stringers and the vertical components are carried by the vertical stirrups, see Fig. 1.

If we assume the stringers to be sufficiently strong, the shear capacity of the beam will be exhausted in the following way: at a certain load level the stirrups will reach the yield stress f_{yw} (it is assumed that the beam is not overreinforced with stirrups). Once the stirrups are yielding, increasing shear force can only be carried by increasing the compression stress σ_c which at the same time rotates to smaller angles

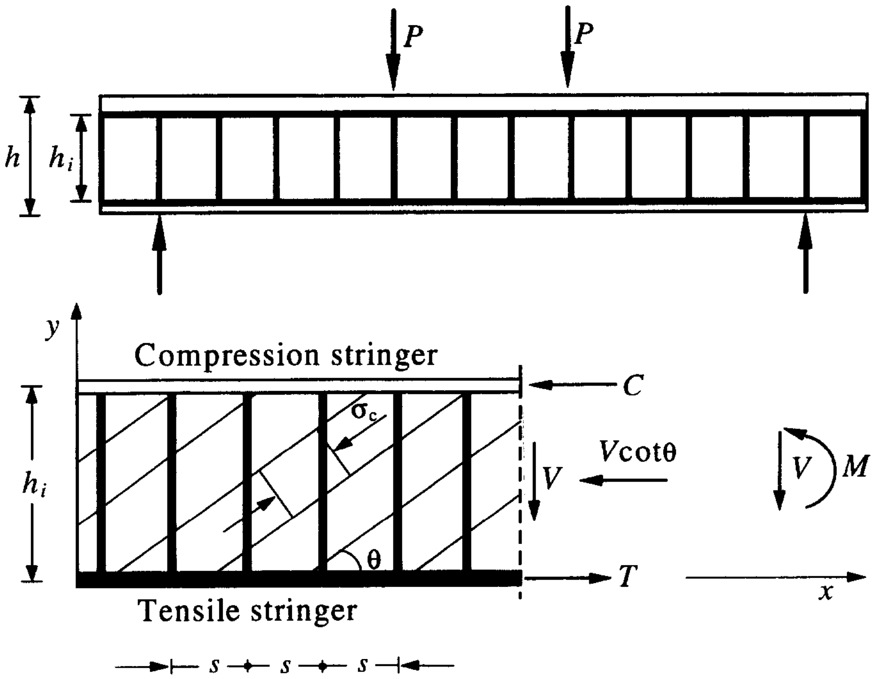

Fig. 1. Diagonal compression field in the web.

θ. The concrete stress σ_c may increase until it reaches the effective compression strength νf_c and the beam finally fails by crushing of the web concrete.

The shear stress, which can be carried if the stirrups are allowed to yield, is

$$\tau = \psi f_c \cot \theta \tag{2}$$

Here, the degree of shear reinforcement ψ is defined as

$$\psi = \frac{A_{sw} f_{yw}}{b s f_c} = \rho_w \frac{f_{yw}}{f_c} \tag{3}$$

A_{sw} being the total sectional stirrup area of one stirrup in a horizontal section and s is the stirrup distance.

The following shear stress can be carried if the concrete is allowed to 'yield':

$$\tau = \frac{\nu f_c}{\tan \theta + \cot \theta} \tag{4}$$

By equalizing eqns (2) and (4) the shear capacity of the beam is found to be

$$\frac{\tau}{\nu f_c} = \begin{cases} \sqrt{\frac{\psi}{\nu}\left(1 - \frac{\psi}{\nu}\right)} & \psi \leq \frac{\nu}{2} \\ \frac{1}{2} & \psi > \frac{\nu}{2} \end{cases} \tag{5}$$

The formulae in eqn (5) are known as the web crushing criterion.

The inclination of the diagonal stress field at failure is given by

$$\tan \theta = \sqrt{\frac{\psi}{\nu - \psi}}, \tan \theta \leq 1 \tag{6}$$

The shear capacity $\tau/\nu f_c$ versus ψ/ν is depicted in Fig. 2.

This lower bound solution is not the exact solution in the whole ψ/ν range. The solution must be modified for small ψ/ν values depending on the shear span. For a discussion of the complete solution, see Ref. 3.

Based on the results of a large number of tests, see Ref. 3 for references, it was found that the effectiveness factor ν with sufficient accuracy could be determined by the simple and well-known empirical formula

$$\nu = 0.8 - \frac{f_c}{200} \quad f_c \text{ in MPa} \tag{7}$$

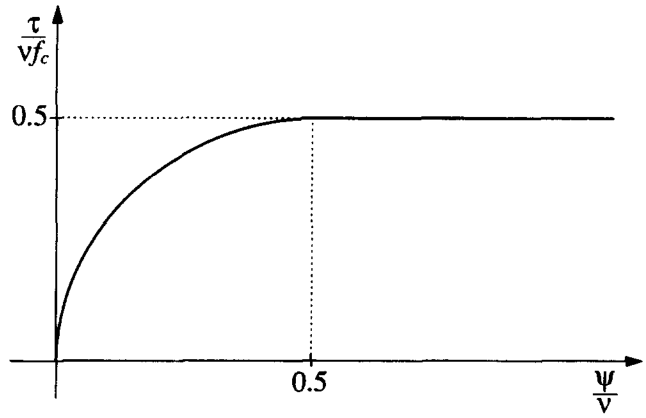

Fig. 2. The web crushing criterion.

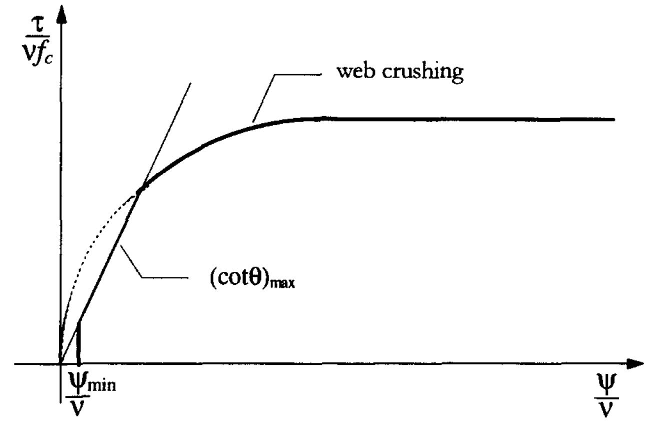

Fig. 3. Code restrictions on maximum cot θ and minimum ψ.

In practice the constant 0.8 has normally been replaced by the conservative value 0.7.

Most of the tests used to confirm the web crushing criterion had relatively large reinforcement degrees and none of them had ψ/ν less than about 0.05.

The web crushing criterion does not agree well with test results when the shear reinforcement degree becomes small. The reasons for this are obvious. When ψ/ν is small, the angle θ must also be small before web crushing can take place. Hence, the diagonal stress field at failure must cross cracks with many different directions. If a diagonal stress field with a compression stress equal to νf_c has to be transferred through these cracks, it would be necessary to supply one or another kind of confinement for instance in the form of extra stirrup reinforcement. Such a confinement is normally not provided. The cracked web region, therefore, may become unstable, resulting in sliding failure along cracks before νf_c is reached. The possibility of sliding in cracks will be further enhanced by the large stirrup distances normally used when dealing with small reinforcement degrees.

When beams with small shear reinforcement degrees are designed by use of the web crushing criterion the shear strength will, for the reasons given above, be overestimated. This problem has been partly overcome in the design codes[5] by introducing a minimum shear reinforcement requirement* and by limiting the choice of cot θ, for instance to cot $\theta \leq 2.5$ for beams with constant longitudinal reinforcement. The latter condition was in fact introduced as a serviceability requirement to reduce the risk of premature yielding but has undoubtedly compensated for the inability of the web crushing criterion to cover cases with small shear reinforcement degrees. The web crushing criterion and the code restrictions are illustrated in Fig. 3.

Referring to Fig. 3, the shear strength of beams with small shear reinforcement degrees is lying somewhere between the straight line and the web crushing criterion.

A more accurate evaluation of the shear strength of such beams can only be carried out by taking crack sliding as well as large stirrup distances into account. This will be demonstrated in the following.

THE CRACK SLIDING MODEL

This section presents the recent developments in the plasticity approach to shear design. The approach is an upper bound approach and is based upon the hypothesis that cracks can be transformed into yield lines, which have lower sliding resistance than yield lines formed in uncracked concrete. The plasticity interpretation of crack sliding was introduced by Zhang in her thesis on the shear strength of non-shear reinforced beams.[6,7]

The model explains in a rational way the shear behaviour of non-shear reinforced beams and it can easily be extended to cover continuous beams, prestressed beams and beams with small shear reinforcement degrees.

* The present Danish code requires a minimum shear reinforcement degree $\phi_{min} = 0.2/\sqrt{10 f_{ck}}$, f_{ck} being the characteristic concrete compressive strength.

Non-shear reinforced beams

Simply supported beams

Consider the case of two symmetrical point loads on a non-shear reinforced beam with rectangular cross-section as shown in Fig. 4. The total depth of the beam is h.

The crack pattern at the state of failure is schematically shown in the figure. The first cracks are normally formed in the region with maximum moment. These cracks are vertical. Then, gradually, diagonal cracks appear in the shear span. These cracks are formed closer and closer to the support, and will, if prolonged, approximately intersect the top face at the loading point.

It is often observed in experiments that shear failure takes place as sliding along the last formed crack. This crack is called the critical diagonal crack. In terms of the plastic theory, the critical diagonal crack is transformed into a yield line.

The transformation of a crack into a yield line has been demonstrated clearly by Muttoni[8] who measured the relative displacements along a crack. When the crack is formed the relative displacement is mainly perpendicular to the crack. When the crack is transformed into a yield line there will be a displacement component parallel to the crack.

In order to calculate the shear strength of the beam, we must answer the following questions: What is the sliding resistance of a crack, and how can we determine the position of the critical diagonal crack?

To answer the first question, we will make the simplified assumption that diagonal cracks are developed following straight lines from the bottom face to the loading point, see Fig. 5. Thus, the diagonal cracks may be described by their horizontal projection x.

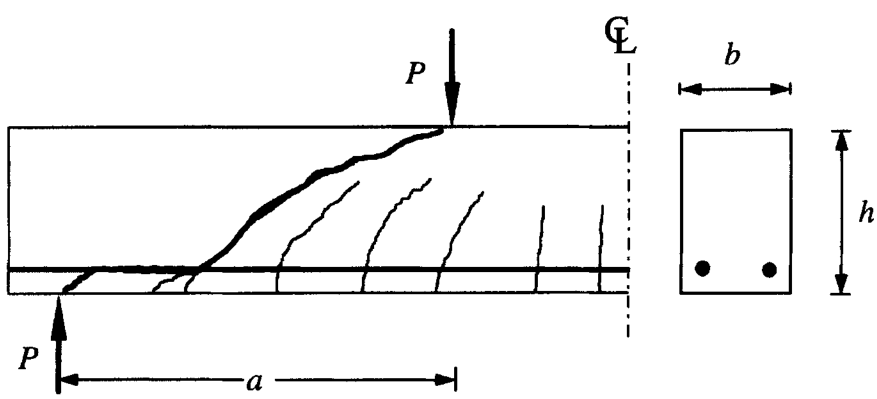

Fig. 4. Non-shear reinforced beam.

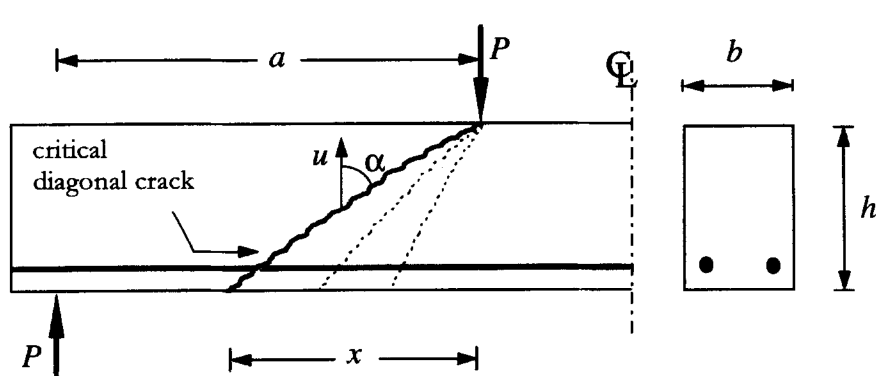

Fig. 5. Beam with idealized diagonal cracks.

Further, we will assume that the beam is overreinforced in the longitudinal direction, imposing the condition on the relative displacement u along the critical diagonal crack to be vertically directed.

It has been shown by Zhang[9] that the modified Coulomb failure criterion may also be applied to a crack. The tensile strength along the crack is naturally zero. By means of a micromechanical model, Zhang has demonstrated that the angle of friction φ and the cohesion c' along a crack may be taken as

$$\left.\begin{aligned} \varphi &\cong 37^\circ \\ c' &= \nu_s c \end{aligned}\right\} \qquad (8)$$

Here, c is the cohesion of uncracked concrete and ν_s is a sliding reduction factor which for normal strength concrete ($f_c \leq 50$ MPa) was found to be

$$\nu_s = \frac{1}{2} \qquad (9)$$

For concrete with relatively weak aggregate materials, where cracks may pass through the aggregate particles resulting in aggregate fracturing, the factor ν_s may be less than 0.5.[10] Here, only beams of normal strength concrete are considered.

For cracks transformed into yield lines the internal work per unit length may now be written as

$$W_I = [\tfrac{1}{2}\,\nu_s \nu_0 f_c (1 - \sin\alpha)]bu \text{ for } \varphi \leq \alpha \leq \pi - \varphi \qquad (10)$$

Here α is the angle between the yield line and the displacement direction. Notice that the constraint on α is due to the fact that crack sliding is a plane strain problem.

Owing to softening it is necessary to introduce an effectiveness factor ν_0, which may be taken to be

$$\nu_0 = \frac{0.88}{\sqrt{f_c}}\left(1 + \frac{1}{\sqrt{h}}\right)(1 + 26\rho),\ \nu_0 \leq 1 \qquad (11)$$

Here f_c is in megapascals, h in metres and ρ is the longitudinal reinforcement ratio.

The factor ν_0 also includes an effect of the absolute scale h and an effect of the longitudinal reinforcement ratio ρ.

Since $\sin\alpha = x/\sqrt{x^2 + h^2}$ the total internal work W_I dissipated in the crack is found to be

$$W_I = \frac{1}{2}\,\nu_s \nu_0 f_c bh\left[\sqrt{1 + \left(\frac{x}{h}\right)^2} - \frac{x}{h}\right]u \qquad (12)$$

Inserting eqn (12) into the work equation, $P_u u$ being the external work, we find the following load-carrying capacity P_u as a function of x:

$$P_u = \frac{1}{2}\,\nu_s \nu_0 f_c bh\left[\sqrt{1 + \left(\frac{x}{h}\right)^2} - \frac{x}{h}\right] \qquad (13)$$

Since this is an upper bound solution, one would expect that eqn (13) should be minimized with respect to x. In that case x should be put equal to a. However, a crack sliding solution corresponding to $x = a$ is only valid if the horizontal projection of the critical diagonal crack is a. This is normally not the case. Thus, x must be determined in another way.

In order to determine the horizontal projection x of the critical diagonal crack, we must determine the load level necessary to form a diagonal crack with the horizontal projection x and the length $\sqrt{h^2 + x^2}$. According to plastic theory, a crack may develop when the effective tensile strength f_{tef} of concrete is reached along the crack path. Hence, referring to Fig. 6, the cracking load P_{cr} may be determined by a moment equation around the loading point O. This is equivalent to a failure mechanism involving rotation around point O. After the formation of a crack the longitudinal reinforcement will be activated to maintain equilibrium.

The cracking load is given by

$$P_{cr} = \frac{\frac{1}{2} f_{tef} b(x^2 + h^2)}{a} \qquad (14)$$

The effective tensile strength, see Ref. 6, may be taken to be

$$f_{tef} = 0.156 f_c^{2/3} s(h) \qquad (15)$$

Here, $s(h)$ is a size effect factor:

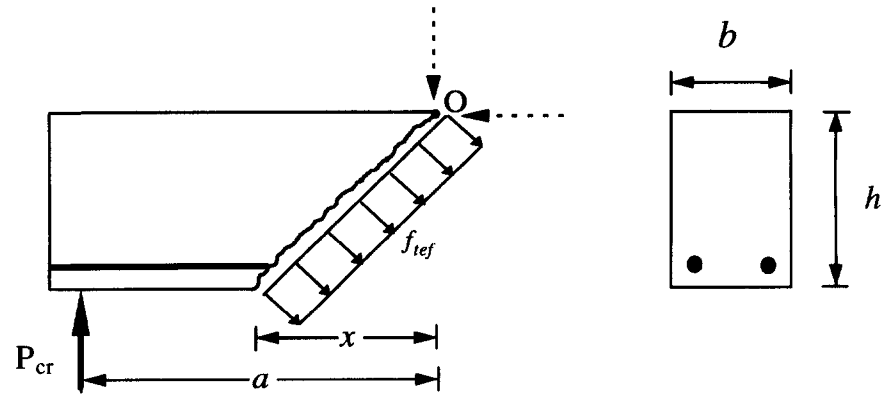

Fig. 6. Stress distribution at formation of a crack.

$$s(h)=\left(\frac{h}{0.1}\right)^{-0.3}, h \text{ in metres} \quad (16)$$

The variation of P_u and P_{cr} versus x is shown in Fig. 7. By following the load increments from zero to failure, Fig. 7 provides a simple and clear explanation of the shear failure mechanism.

At lower load levels only steeper cracks can be formed. These cracks are indicated in Fig. 5 as the dotted cracks. Sliding failure along these cracks cannot take place since their sliding strength P_u is larger than the load needed to form them. This is true as long as the P_{cr}-curve is lying below the P_u-curve. Sliding failure along a crack immediately after it has been formed is only possible if the load needed to form the crack equals the sliding strength of the same crack. Thus, the shear strength of the beam as well as the horizontal projection of the critical diagonal crack is determined by the intersection point of the curves representing P_u and P_{cr}. This is illustrated in Fig. 7.

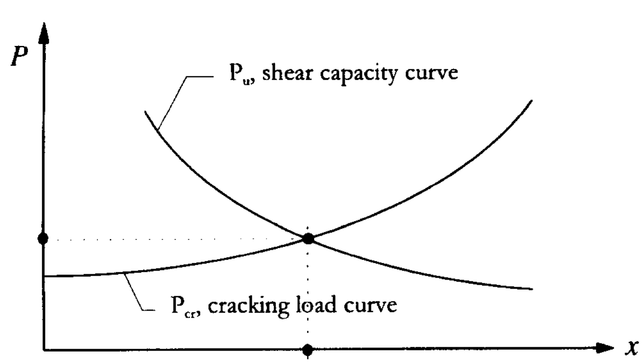

Fig. 7. Cracking load and shear capacity versus x.

The procedure of calculation is as follows: first the equation $P_u = P_{cr}$ is solved with respect to x. Thereupon x is inserted into eqn (13) to determine the shear capacity.

If $x > a$ then a must be used when calculating the shear capacity. Further, sliding is only possible for $x/h \geq 0.75$. This is due to the constraint, imposed by the normality condition, on the angle between the relative displacement and the yield line.

Some additional effects and conditions due to the fact that cracks are formed with finite distances have been discussed by Zhang[6] to which the reader is referred.

Simplified calculation

The calculation may be strongly simplified if the correct formula for the internal work (eqn (12)) is replaced with the following approximation, see Refs 11 and 12:

$$W_1 = 2\,\frac{\tau_c}{x/h}\,bhu \quad (17)$$

where

$$\tau_c = 0.059\nu_0 f_c \quad (18)$$

Using this approximation, P_u turns into

$$P_u = 2\,\frac{\tau_c}{x/h}\,bh \quad (19)$$

Now, by equalizing eqns (14) and (19), we obtain the following cubic equation rendering x/h:

$$\left(\frac{x}{h}\right)^3+\frac{x}{h}-4\frac{\tau_c}{f_{tef}}\frac{a}{h}=0 \quad (20)$$

Thus, the shear strength may be found by inserting the solution of eqn (20) into eqn (19). Alternatively, the shear strength may also be expressed by the average shear stress $\tau_u = P_u/bh$:

$$\tau_u = 2\frac{\tau_c}{x/h} \quad (21)$$

Solutions for prestressed beams and beams with uniform loading may be found in Refs 6 and 7. Application of the model to prestressed hollow-core slabs is explored in Ref. 13. For all the cases treated good agreement with experiments has been obtained.

Although rather rough, the suggested method of determining the cracking load seems to work well. Improvements on this point require a fracture mechanics approach. Gustafsson[14] determined the cracking load by the Hillerborg model using FEM-calculations. He also observed that the position of the crack and the load-carrying capacity may be determined by the intersection point of two curves as shown in Fig. 7.

Continuous beams

The crack sliding model presented is also capable of treating continuous beams. This will be demonstrated for the case of a point load in the middle of a beam with rectangular cross-section, fixed in one end and simply supported in the other end, see Fig. 8. The beam is assumed to have the same constant reinforcement at top and bottom. As before, we assume overreinforcement in the longitudinal direction. Thus, no moment redistribution will take place prior to failure.

The crack pattern, observed in tests, is schematically shown in the figure. Sliding failure will take place in a diagonal crack emerging from the fixed support due to the relatively larger shear force in this region.

The shear strength, expressed as the average shear stress at the fixed support $\tau_u = V/bh$, is identical to eqn (21). This is easily verified by writing down the work equation with Vu as the external work.

The cracking load, however, will differ from the case of a simply supported beam due to the contribution from the bending moment at the fixed end. This moment is considered as an external load. The cracking load may be found by a moment equation around point S, see Fig. 8, or equivalently by considering a rotation mechanism around point S.

Expressions for the cracking load for this and other cases of statically indeterminate beams may be found in Ref. 11. Here we only state the cubic equation rendering the x/h.

$$\left(\frac{x}{h}\right)^3+\frac{x}{h}-\frac{24}{11}\frac{\tau_c}{f_{tef}}\frac{a^*}{h}=0 \quad (22)$$

Notice that the shear span in this case is denoted a^*.

With eqns (20) and (22) in mind, we may pose the important question: What is required

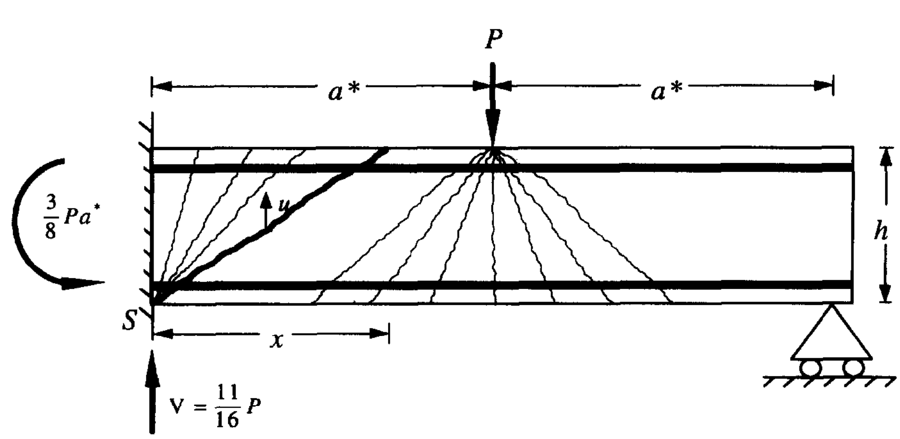

Fig. 8. Half of two-span continuous beam.

for the two types of beam considered to have the same shear strength? If we assume that the material properties are identical, then the condition to be fulfilled is that the solution of eqns (20) and (22) must be identical. We see that this condition requires the following relation:

$$\frac{a}{h} = \frac{6}{11}\,\frac{a^*}{h} \tag{23}$$

which is equivalent to the condition

$$\frac{M_1}{Q_1 h} = \frac{M_2}{Q_2 h} \tag{24}$$

Here, M_1 and Q_1 denote the internal forces at the loading point of the simply supported beam, and M_2 and Q_2 denote the internal forces at the fixed support for the beam shown in Fig. 8.

It is interesting to ascertain that eqn (24) is in fact an empirical rule (derived from test results) given by Leonhardt *et al.*[15] for the purpose of answering the posed question.

Thus, the model presented has reproduced an old empirical rule. This is indeed a triumph of plastic theory.

Lightly reinforced beams

Smeared approach

The crack sliding problem is also met in beams with small shear reinforcement degrees. Such beams are termed lightly reinforced beams. In what follows, we shall demonstrate that the crack sliding model is capable of treating lightly reinforced beams as well.[10]

Again, we consider the case of two symmetrical point loads on a simply supported beam with rectangular cross-section, see Fig. 9. The beam is assumed to be overreinforced in the longitudinal direction.

Sliding failure in a diagonal crack with the horizontal projection x is considered. For the time being, we consider the stirrups to be closely spaced in the web (smeared approach). The shear capacity, as a function of x, may then immediately be written down:

$$P_u = \left(\frac{2\tau_c}{x/h} + \psi f_c \frac{x}{h} \right) bh \tag{25}$$

which, rewritten using an average shear stress, takes the form

$$\tau_u = 2\,\frac{\tau_c}{x/h} + \psi f_c \frac{x}{h} \tag{26}$$

The cracking load may be calculated using eqn (14), thereby neglecting the influence of the shear reinforcement on the cracking load.

The cracking load curve and the shear capacity curve are schematically shown in Fig. 10. Unlike the case of a non-shear reinforced beam, the shear capacity curves now display a minimum value. Thus, if the cracking load curve intersects the shear capacity curve for a value of x smaller than that corresponding to the minimum value, the load-carrying capacity will correspond to the intersection point. If, however, the minimum value is reached at an x-value smaller than that corresponding to the intersection point, the load-carrying capacity will be determined by the minimum value of the shear capacity curve. The two situations are illustrated in Figs 10(a) and (b).

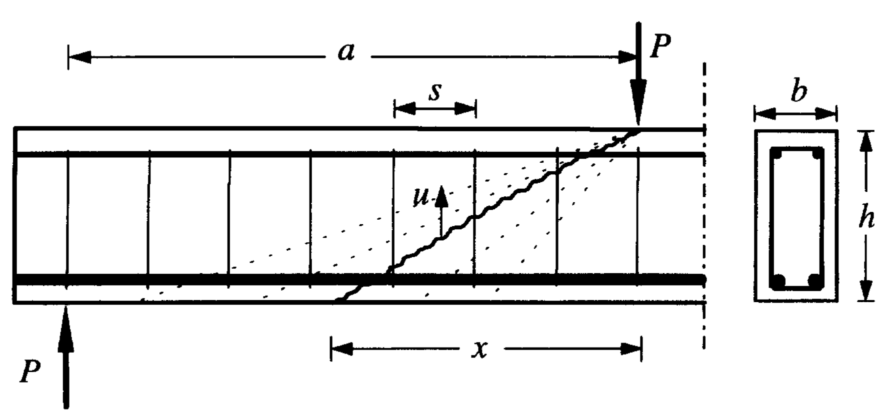

Fig. 9. Simply supported lightly reinforced beam.

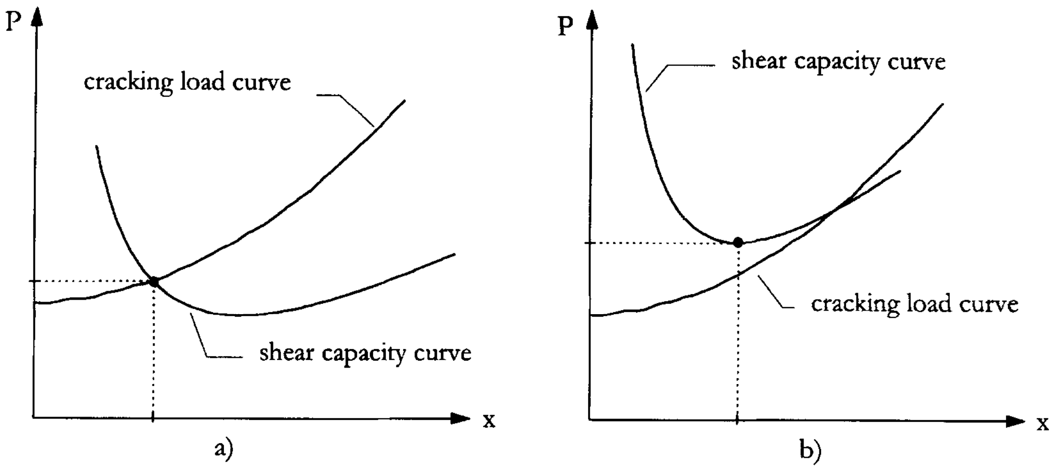

Fig. 10. Cracking load and shear capacity versus x for beam with shear reinforcement.

In the first situation, failure will take place immediately after the critical crack has been formed, whereas the load in the latter situation must, when the critical crack has formed, increase until it reaches the minimum value of the shear capacity curve before failure takes place. It turns out that the situation illustrated in Fig. 10(b) is already topical at very small shear reinforcement degrees, see Ref. 10. In the following, only the situation in Fig. 10(b) will be treated in detail.

The minimum value of the shear capacity curve is easily determined by minimizing eqn (26) with respect to x/h. The result is

$$\frac{x}{h} = \sqrt{\frac{2\tau_c}{\psi f_c}} \tag{27}$$

and

$$\frac{\tau_u}{\tau_c} = \sqrt{\frac{8\psi f_c}{\tau_c}} \tag{28}$$

Inserting eqn (18) into the right-hand side of eqn (28) we find the following simple expression:

$$\frac{\tau_u}{\tau_c} = 11.64\sqrt{\frac{\psi}{v_0}} \tag{29}$$

Effect of finite stirrup distances

So far, the effect of finite stirrup distances has been neglected. However, when dealing with large stirrup distances, the solution derived may lead to quite unsafe results. To take into account the effect of finite stirrup distances, the shear capacity curve must be drawn by counting the number of stirrups crossing the cracks. If the stirrup distance is s and the shear reinforcement degree is ψ, each stirrup crossed adds $\psi f_c s/h$ to the shear capacity τ. An example of a shear capacity curve constructed as described is shown in Fig. 11. The curve will of course be discontinuous.

The shear capacity curve corresponding to the smeared stirrup approach has also been shown in the figure (the dotted line). Notice that the two curves touch whenever x is a multiple of s (i.e. $x = s$, $2s$, $3s$, etc.).

In cases where the minimum value of the two shear capacity curves is localized at the same value of x, which hereby must assume a value given by a multiple of s, the effect of finite stirrup distances may be incorporated in the solution in eqn (29) in a very simple way; namely by deducting the contribution $\psi f_c s/h$ from eqn (29). This is illustrated in Fig. 11 where the shear capacity drops from point A to point B. (The figure illustrates a situation where the minimum value of the two shear capacity curves, at A and B, is localized at $x = s$).

The solution taking into account finite stirrup distances will hereby take the simple form

$$\frac{\tau_u}{\tau_c} = 11.64\sqrt{\frac{\psi}{v_0}} - 16.95\,\frac{\psi}{v_0}\,\frac{s}{h} \tag{30}$$

Naturally, it is more an exception than a rule that the minimum values of both shear capacity curves are localized at the same value of x. It

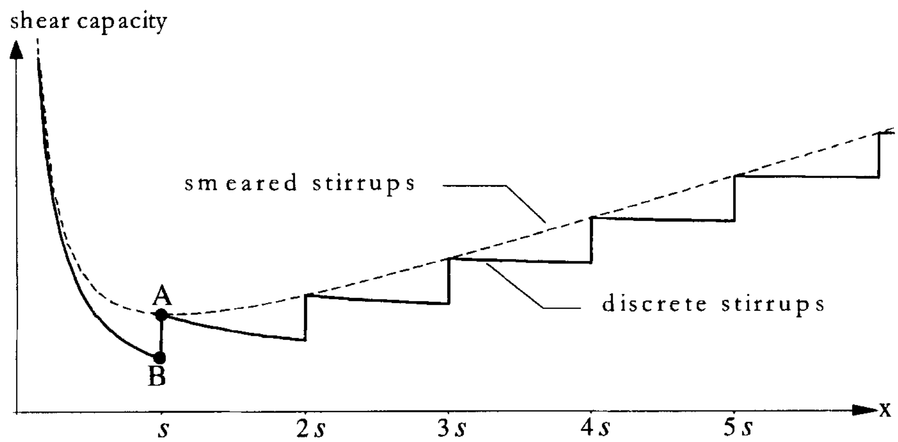

Fig. 11. Shear capacity curves according to smeared approach and discrete approach.

may be shown, however, that the difference between the minimum values of the two curves will never exceed a value corresponding to $\psi f_c s/h$. Thus, the solution in eqn (30) is generally a safe and reasonable approach to take into account the effect of finite stirrup distances.

Owing to the normality condition the solution derived is valid only when x/h given by eqn (27) is larger than 0.75. By requiring $x/h \geq 0.75$, the following condition for the validity of eqn (30) is found:

$$\frac{\psi}{v_0} \leq 0.21 \qquad (31)$$

When $\psi/v_0 > 0.21$ the shear capacity is calculated by inserting $x/h = 0.75$ into eqn (26) rendering a linear variation of the shear capacity versus ψ.

When incorporating the effect of finite stirrup distances in the case where $\psi/v_0 > 0.21$, the shear capacity formula reads:

$$\frac{\tau_u}{\tau_c} = \begin{cases} 11.64\sqrt{\dfrac{\psi}{v_0}} - 16.95\dfrac{\psi}{v_0}\dfrac{s}{h} & \dfrac{\psi}{v_0} \leq 0.21 \\ 2.67 + 16.95\dfrac{\psi}{v_0}\left(0.75 - \dfrac{s}{h}\right) & \dfrac{\psi}{v_0} > 0.21 \end{cases} \qquad (32)$$

Notice that if $s/h > 0.75$ when $\psi/v_0 > 0.21$ the stirrups will give a negative contribution. This result, which of course is not true, is, however, not of practical interest. Designers usually choose stirrup distances to be less than $0.75h$, especially when $\psi/v_0 > 0.21$, which is a relatively large shear reinforcement degree.

In cases where x/h given by eqn (27) is larger than the shear span a, an additional condition must be introduced. Thus, there is a lower limit of ψ/v_0 below which eqn (30) cannot be used. The reader is referred to Ref. 10.

The solution in eqn (32) is shown in Fig. 12. The reduction due to the finite stirrup distances appears to be very drastic for large stirrup distances combined with large shear reinforcement degrees.

However, such reductions will not appear in reality if the beams are designed using normal practice. Designers usually choose to operate with one or two types of stirrup. ψ is thus increased by decreasing s. In such cases, the reduction will be constant ($= A_{sw}f_{yw}/bh$).

For comparison, the web crushing criterion has also been drawn in Fig. 12. To use the same (non-dimensional) nominal shear stress, the web crushing criterion given by eqn (5) must be multiplied by h_i/h and divided by τ_c. By drawing the web crushing curve in Fig. 12, it has been assumed that $h_i/h = 0.8$ (which is a typical value). Further, it has been assumed that $v = v_0$. This is obtained for the following typical values: $f_c = 25$ MPa, $h = 300$ mm and $\rho = 1.4\%$.

In Fig. 12 we notice that even in the case of closely spaced stirrups ($s/h = 0$) crack sliding should be taken into account for shear reinforcement degrees up until $\psi/v_0 \cong 0.25$.

Strictly speaking, the crack sliding criterion should always be applied if the corresponding shear strength lies below the one belonging to the web crushing criterion. However, as stated above, large reinforcement degrees will (normally) be accompanied with closely spaced stirrups, which makes the assumption of smeared stirrups more realistic. Thus, the trans-

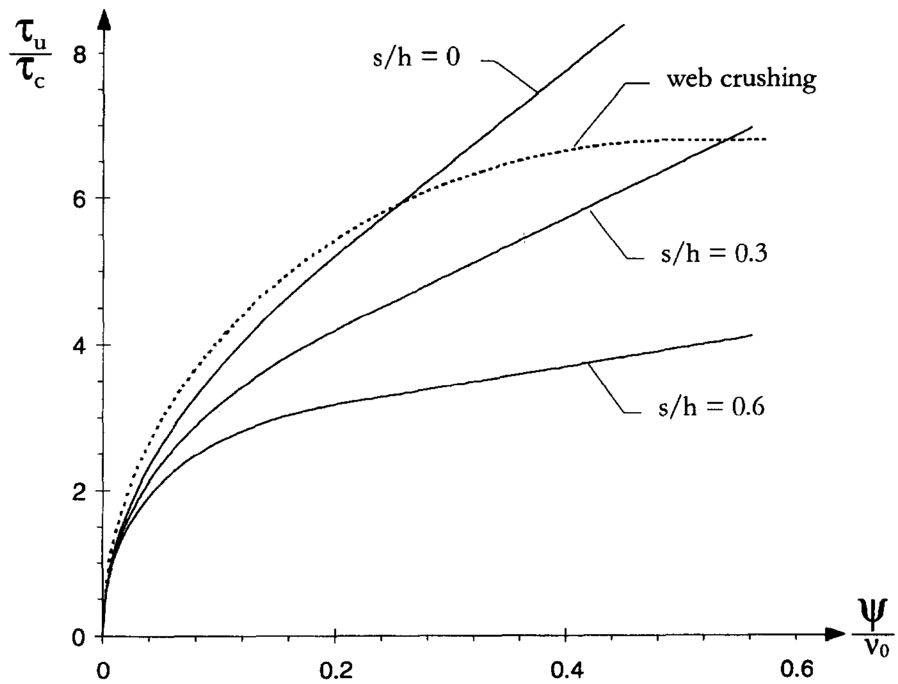

Fig. 12. Crack sliding criterion and web crushing criterion versus shear reinforcement degree.

ition point between crack sliding and web crushing may, in practice, be taken at the shear reinforcement degree $\psi/\nu_0 = 0.25$.*

Comparison with experiments

The solution in eqn (32) has been compared with some test series. Only a few of the test series reported in the literature had varying shear reinforcement degree with constant stirrup distance. One small series was reported by Regan & Placas in Ref. 17.* The results are shown in Fig. 13. It can be seen that the test results are in good agreement with the crack sliding solution.

In Fig. 14 the results of 106 beam tests from five test series have been compared with the calculated shear strength. It can be seen that the test results are in fairly good agreement with the theory. The mean value of the ratio $\tau_{test}/\tau_{theory}$ is 0.97 and the standard deviation is 0.17.

The concrete strength and the shear reinforcement degree of these tests varied between:

$f_c \in [13\text{–}48\ \text{MPa}]$ (average strength 30 MPa)

$\psi \in [0.006\text{–}0.174]$ (average ψ 0.043)

Crack sliding solutions for lightly reinforced beams subjected to uniform loading and for continuous beams may be found in Ref. 10. For these cases, good agreement with experiments was found too.

THE EFFECTIVENESS FACTORS

The use of reduced concrete strengths through a multiplication with an effectiveness factor on f_c has been, and still is, a subject of debate among researchers. Researchers with different attitudes to practical engineering design take different stands in the debate. But, no matter what the opinions may be, the necessity of working with effectiveness factors is indispensable when the rigid and perfectly plastic material model is applied.

The first effectiveness factor was introduced in a discussion on the web crushing criterion.[22] Based on a test series of Leonhardt & Walther[23] it was found that $\nu = 0.85$. The interpretation of the effectiveness factor at that time was that only the part of the web concrete confined by the stirrups is active (the concrete cover must remain unstressed to match equilibrium requirements). This interpretation thus

*This agrees with the experimental observations in Ref. 16 where yielding of stirrups at failure was not detected when ψ became larger than about 0.2. This indicates that crack sliding did not take place.

*Information concerning the stirrup distances was not given in Ref. 17 but provided by a personal communication with the authors.

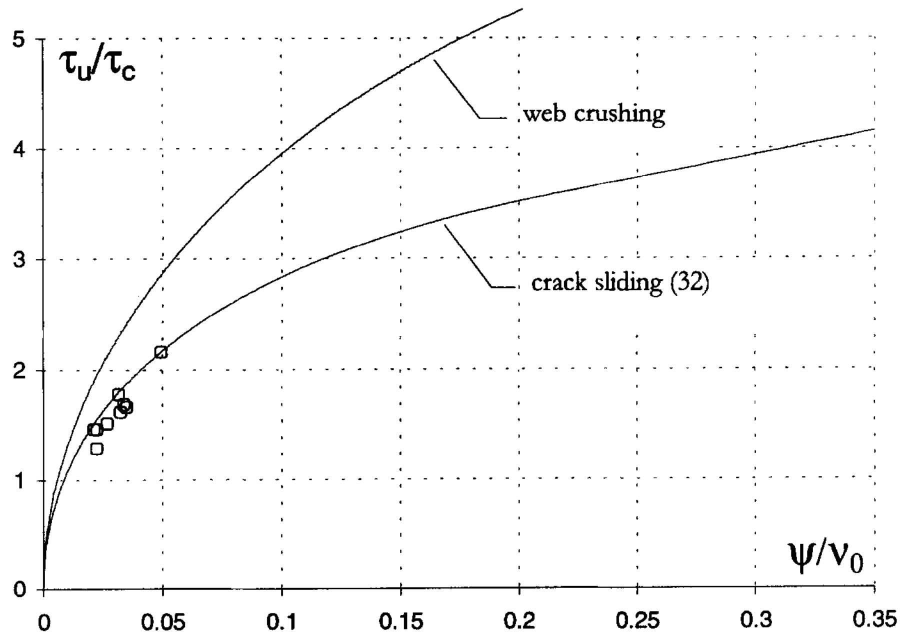

Fig. 13. Tests by Regan & Placas[17] ($s/h = 0.5$ and average value of $\nu/\nu_0 = 0.95$).

addressed the 'problem' to the geometry of the stirrups and not to the properties of concrete! Much has been learned since then.

The last decades of research on the properties of concrete and on the behaviour of concrete structures has brought us to a better understanding of the physical meaning of the effectiveness factors and why different problems require different values.

At present it is believed that the effectiveness factors take into account the following main effects:

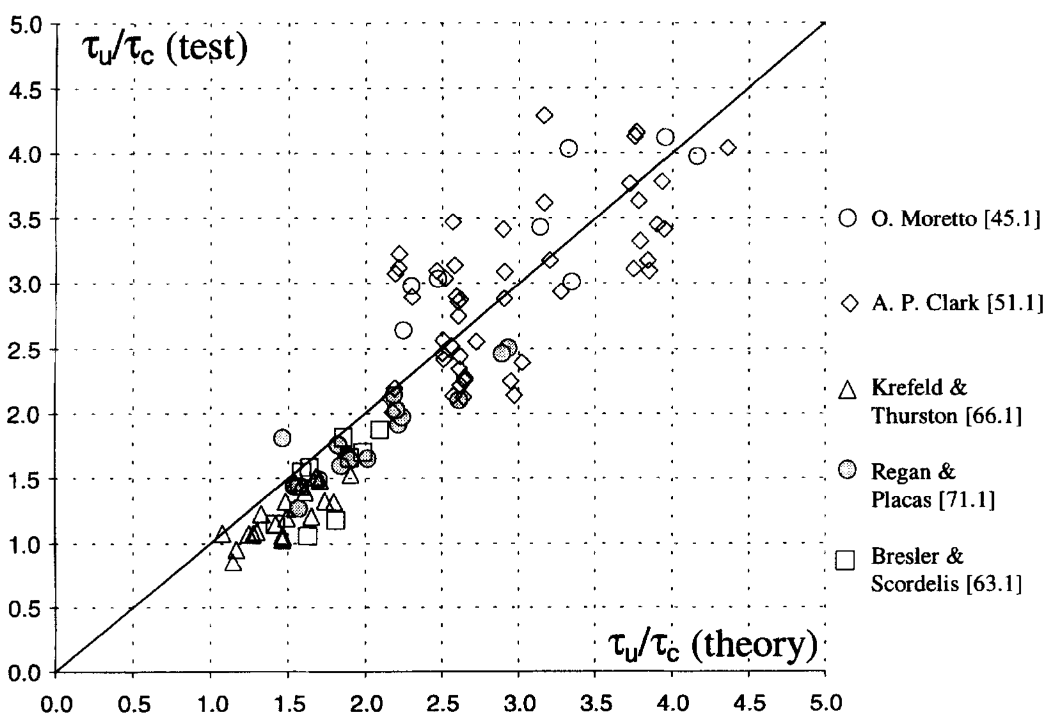

Fig. 14. Comparison of theory with 106 test results: ○, Moretto;[18] ◇, Clark;[19] △, Krefeld & Thurston;[20] ●, Regan & Placas;[17] □, Bresler & Scordelis.[21]

1. the softening post-peak behaviour of unconfined concrete;
2. cracking of concrete;
3. local damage of concrete prior to failure due to stressed transverse reinforcement.

Softening. The effect of softening will enter in all problems with inhomogeneous strain fields. Owing to softening the stresses along a failure surface do not at every point assume the maximum value at the same time, see the illustrations in Fig. 15. Thus, instead of using f_c, we must deal with a reduced strength which expresses the average stress maintained in the fully developed failure surface. The effect of softening is more pronounced for high values of f_c and for large structures. (Therefore ν_0 decreases with increasing f_c and h.)

Cracking. The effect of cracking, which we have been dealing with in this paper, will enter in all problems involving crack sliding failure. The strength reduction due to cracking has been taken as a constant ($\nu_s = 0.5$). It is to be expected that the validity of this assumption depends upon the roughness of the crack surface as well as the width of the crack. The roughness of the crack surface may possibly be taken into account by expressing ν_s as a function of for instance a relative strength and a relative size of the aggregate materials. The influence of the crack width on ν_s may be taken into account by the parameters which directly or indirectly

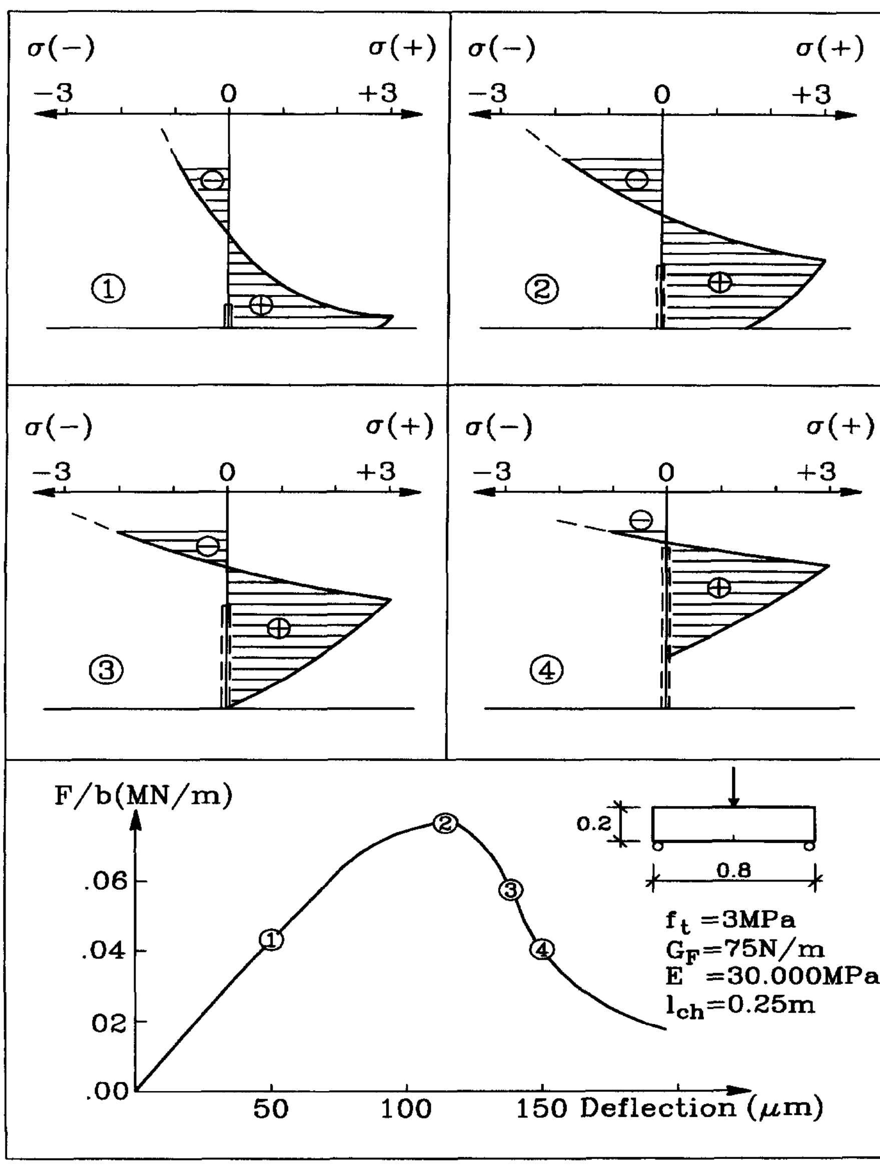

Fig. 15. Response to flexural failure of a beam of softening material.[24]

control the crack width, e.g. the reinforcement ratio ρ.

Local damage. Local damage of concrete prior to failure reduces the area of the final failure surface or the area carrying the load, thus resulting in a reduced load-carrying capacity. The local damage may take the form of cover spalling or local punching around the reinforcement bars. These damages are due to the bursting stresses transferred from the stressed reinforcement bars to the concrete. Studies on the effect of local damage may be found in Ref. 25 where effectiveness factors for this effect may also be found. Figure 16 shows examples of local damage.

While the individual effects are now much better understood, it is still an open question as to how to find the resulting effectiveness factor. There is some evidence that the resulting formula may be obtained by simple multiplication of the individual effectiveness factors.

Consider first homogeneous strain fields. If a yield line in unreinforced concrete is formed in a crack the effectiveness factor will be $\nu_{slid} = \nu_s$. If the concrete is reinforced so that there will be stressed transverse reinforcement in the yield line, the effectiveness factor will be ν_{burst}. If the yield line in the latter case is formed in a crack, evidently the effectiveness factor is $\nu_{slid}\nu_{burst}$.

When dealing with inhomogeneous strain fields in unreinforced concrete and with no yield lines in cracks, there is an effectiveness factor ν_{soft} due to softening. An example is a corbel with small shear span so that sliding in cracks cannot take place. In this case ν_{soft} is the only effectiveness factor and it may be found using eqn (11). When sliding in cracks may take place, for instance for larger shear spans, it has been demonstrated in this paper that the effectiveness factor may be taken as $\nu_{slid}\nu_{soft} = \nu_s\nu_0$, ν_s being 0.5 and ν_0 determined by eqn (11).

In this context the formula in eqn (7) valid for the effectiveness factor for shear reinforced beams with large reinforcement degrees belongs to the type ν_{burst}. For such beams crack sliding may be disregarded and it seems that no ν_{soft} is present. One may hope that this is a general trend, i.e. softening effects may be neglected when we are dealing with concrete reinforced with a uniform reinforcement in two directions and possibly a concentrated reinforcement normally placed along the boundaries.

Thus, there is some evidence that the most general formula for the effectiveness factor is

$$\nu = \nu_{slid}\nu_{soft}\nu_{burst} \tag{33}$$

One or more terms may be missing as explained above. For normal strength concrete $\nu_{slid} = \nu_s$ may always be taken as 0.5. For non-shear reinforced beams and lightly reinforced beams in shear ν_{burst} may be neglected and ν_{soft} may be determined by eqn (11). For beams with very small shear spans only ν_{soft} remains. It may be determined by eqn (11). For beams in shear with large reinforcement degrees both ν_{slid} and ν_{soft} may be neglected and ν_{burst} may be calculated by eqn (7).

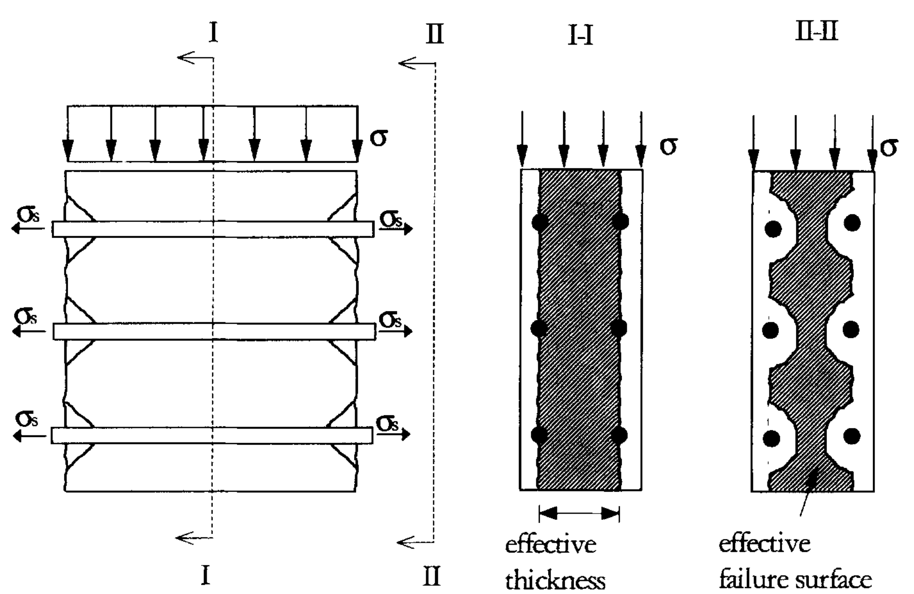

Fig. 16. Local damage due to bursting stresses.

This scheme is possibly somewhat oversimplified, but it is in agreement with the large number of experimental findings as they have been collected up to now.

However, formulas like eqn (33) can never be complete. The anchorage of the reinforcement along the faces may impose restrictions on the compressive stresses which can be carried. For shear reinforced beams this interpretation of v might be more to the point than an v_{burst} effect. In fact, most of the tests on shear reinforced beams were done with plain bars giving rise to much less bursting forces than deformed bars.

So that is where we are today. Back to the 69-reasoning.

Much more research is needed in this field. And this research should be done irrespective of a favourable or unfavourable view on plastic theory. In any FEM code such information is needed. The only effect such a program may deliver, at least in the near future, is the v_{soft} effect.

CONCLUDING REMARKS

The recent years of code implementation of plastic methods in many countries has clearly demonstrated that the plastic theory for reinforced concrete is indeed a powerful tool for practical design.

Plastic theory provides simplicity and freedom of choice in the design phase. It supplies a clear picture of the failure mechanism in the structure considered.

In the future the use of sophisticated FEM programs will undoubtedly increase, as more and more large-scale complex structures will be built. The engineer will thus be placed in a position where he/she will find it difficult, if not impossible, to form a general view of the results of complex FEM analyses. The consequence of undetected errors may be catastrophic (one scary example is the SLEIPNER-platform disaster). More than ever, the engineer will need to mobilize his/her commonsense and familiarise himself/herself with the behaviour of the analysed structure.

In this context, plastic methods may serve as a tool providing fast estimates, based on which the engineer can judge the output of a detailed FEM analysis.

This may be one of the main roles the plastic methods will play in the future.

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