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Effect of nonlinear response of concrete on its elastic modulus and strength

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Abstract

The purpose of this paper is to investigate the effect of nonlinear response of concrete on the relationship between modulus of elasticity at static and dynamic loading as well as on its strength. The obtained relationships are based on the thermofluctuation strength theory coupled with a nonlinear stress–strain material model. From the corresponding equations it was found that the ratio of the static to the dynamic modulus of elasticity depends on the strength of concrete, its temperature, rate of loading. Also it was confirmed that the dynamic modulus is greater than static modulus of elasticity. These equations explain the influence of the value of applied stress on the value of the static or dynamic modulus of elasticity. Comparative study shows substantial agreement with existing experimental results and the general equations given in standard BS 8110, Part 2:1985, and ACI documents. Based on the obtained relationships new methods for evaluating the static modulus of concrete and its strength from the results of dynamic tests are described subsequently.

Keywords: Elastic moduli; Strength; Nonlinear deformation; Nonlinear nondestructive methods

1. Introduction

Classical evaluations of mechanical properties in accordance with Hook's law demonstrate the ability of a homogeneous, isotropic, elastic material to be deformed in direct proportion-linearly-to extension or compression [1]. However, beginning with experiments by James Bernoulli [2], and until the present careful investigations of different materials (stone, concrete, wood, glass, metal etc.) show the nonlinear dependence between stress and strain even at infinitesimal values of deformation at quasi-static tests and dynamic tests based on the stress-wave propagation [3].

The presence of nonlinear dependence has been derived from basic provisions of the modern theory of solid bodies, and is stipulated by the nature of interatomic forces [4,5]. Different flaws in structure (dislocations, micropores, and microcrevices) have a significant influence and predetermine the nonlinear behavior of real materials under loading. For instance, the examination of concrete specimens subjected to increasing uniaxial compression, using microscopes and X-ray technique, showed that bond cracks exist in concrete even before any loading. Starting at 30% of ultimate load, bond cracks increase rapidly in length, width and number [6,7]. Troxell and Davis showed higher values of Poisson's ratio at low stress levels at uniaxial compression [8]. The beginning of an increase in Poisson's ratio indicates significant increase in bond cracking [9].

Dominant influence on nonlinearity of deformation such as flaws in the structure of concrete like micropores, microcracks and other discontinuities (defects) is established from results of ultrasonic, metric and nuclear measurements on saturated and dried to constant weight

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specimens [10]. It was recognized that in dry specimens of concrete the velocities of longitudinal and shear waves (or dynamic modulus of elasticity) increase at stress level below (25–30)% of ultimate load, and start to decrease above this level. In general the changes of modulus of elasticity of concrete, or nonlinear parameters, are typically 2–3 orders of magnitude greater in comparison with those of crystalline or polycrystalline materials, in spite of a crystal structures, which make up the main mass of solid phase in concrete. It shows a high sensitivity of a nonlinear parameter to the element of structure, which dimensions are much smaller than the wavelength at dynamic tests of concrete [11].

Therefore, concrete subjected to increasing uniaxial compression reflects the reaction of its structure on different values of external forces and indicates the nonlinearity of deformation. It means that the fracture of concrete during loading does not occur as a critical event. It is rather the result of a process of nucleation, accumulation, and development of damage elements. In accordance with thermofluctuation theory, the fracture is a kinetic or time dependent phenomenon [12]. At the atomic–molecular level this process is controlled by events of the breaking of interatomic bonds by thermal fluctuations. At the macroscopic level the fracture of stressed concrete proceeds by a process that develops in the body from the moment of load application, while the fracture itself is the final act in this process.

The extremely complex nature of concrete has forced investigators to use classical (linear) approach for evaluating modulus of elasticity [13–18].

Based on the results of these researches and thermofluctuation theory, the object of this paper is to discuss a method for evaluating the effect of nonlinear behavior of concrete on the relationship between modulus of elasticity at static and dynamic loading as well as on its strength.

2. Theoretical background

Thermofluctuation theory leads to a general relation between strength σ_r and the time τ to failure of solids under loading [12]

$$\sigma_r = \frac{U_0}{\gamma} + \frac{kT}{\gamma}\ln\tau_0 + \frac{kT}{\gamma}\ln\frac{1}{\tau}, \tag{1}$$

where U_0 is the activation energy of destruction, γ is the structurally-sensitive coefficient, k is the Boltzmann constant, T is the temperature, τ_0 is the period of interatomic vibration and equal 10^{-13} s, $kT = 404\times10^{-23}$ J (2.44 kJ/mole) at the usual temperature of test (T = 293 K).

The following Eq. (2) is obtained for the simplest case of stress growth at constant rate $\dot\sigma$ taking into account the principle of summation of damages under loading [19,20]

$$\ln\frac{1}{\tau} = \ln\frac{\gamma\dot\sigma\tau_0\exp(U_0/kT)+1}{kT\exp(U_0/kT)} \tag{2}$$

In an extremely wide range of real values of U_0, γ, and not limiting the small loading rate ($\dot\sigma = 10^{-7}$ MPa/s), it is possible to neglect one in the numerator of Eq. (2): for concrete $U_0 \sim 124$ kJ/mole; $\gamma \sim 3$ kJ/(mole MPa) at compressive loading [11,21]. Rearranging, Eq. (1) can be rewritten in closed form [19,20] as

$$\sigma_r \cong \frac{1}{\gamma}\left[U_0 + kT\ln\left(\tau_0\frac{\gamma}{kT}\right)\right] + \frac{kT}{\gamma}\ln\dot\sigma \tag{3}$$

It was shown also that the linear dependence σ_r ($\ln\dot\sigma$) extrapolates in an unforced way to a pole on the abscissa, i.e.

$$\ln\dot\sigma_0 = -\left[\frac{U_0}{kT} + \ln\left(\tau_0\frac{\gamma}{kT}\right)\right] \tag{4}$$

For a number of different cases of fracture of concrete (different concretes, different forms of stress, simply different investigations in different countries) a fairly reliable value of $\log\dot\sigma_0 = -8.5$ ($\dot\sigma_0$ in MPa/s) was found. This value is not incidental because corresponds to the stable value of activation energy for concrete $U_0 \sim 120$ kJ/mole, which is close to the dissociation energy of the silicon–oxygen bonds in accordance with hydrolytic mechanism [22]. Therefore, with allowance for Eqs. (3) and (4) the general Eq. (1) is transformed to the form

$$\sigma_r = \frac{kT}{\gamma}\ln\left(\frac{\dot\sigma}{\dot\sigma_0}\right) \tag{5}$$

Now it is possible to derive the relationship between the structurally-sensitive coefficient γ and the change of the modulus of elasticity of concrete under loading, i.e. nonlinear parameter. Indeed. Within a wide range of strain (10^{-8}–10^{-4}), the relationship between stress σ and strain ε of different materials, including those that are brittle, can be described as a quadratic parabola. This nonlinear function (1) demonstrates qualitatively and quantitatively the response from the largest to the smallest deformations, and (2) the value of modulus of elasticity extrapolated to zero stress (E_0) closely agrees with the dynamic modulus determined from longitudinal vibrations with amplitudes of 10^{-6} [3]. Therefore the dependence between stress acting on the body and strain can be presented as

$$\sigma = E_0\varepsilon - b\varepsilon^2, \tag{6}$$

where constant "b" is the nonlinear parameter. One can see that for the nonlinear model (6) the modulus of elasticity depends on strain $E = \mathrm{d}\sigma/\mathrm{d}\varepsilon = E_0 - 2b\varepsilon$, and the value of "$b$" can be normalized to the value of E_0

$$b = (E_0 - E)/2\varepsilon = E_0(1 - E/E_0)/2\varepsilon \tag{7}$$

$$b/E_0 = (E_0 - E)/2E_0\varepsilon = \mathrm{d}E/2\mathrm{d}\sigma \tag{7a}$$

By assuming equal distribution of external load, all interatomic connections are strained equally, and the average stress σ acting on the body is equal to the stress σ_a acting on the atom. Under the action of stress σ_a, which is less than the ultimate value, the potential energy $U(\sigma_a)$ is proportional to the volume of atom (W_a), and in accordance with (6) becomes [11]

$$U(\sigma_a) = U_0 - W_a\sigma_a b/E_0 \tag{8}$$

where U_0, W_a is the initial potential energy, and volume of atom ($W_a \sim 10^{-2}\ \text{nm}^3$), respectively.

On the other hand, it was shown [12,23] that the potential energy decreases linearly with applied stress as

$$U(\sigma_a) = U_0 - \gamma\sigma_a \tag{9}$$

Then from (7)–(9) follows

$$\gamma = W_a b/E_0 = W_a(E_0 - E)/(2\varepsilon E_0) \sim W_a(E_0 - E)/2\sigma \tag{10}$$

or

$$\gamma \sim W_a \frac{1}{2}\frac{\partial E}{\partial \sigma} \tag{10a}$$

Thus, Eqs. (10) and (10a) show the relationship between the structurally sensitive coefficient "γ" from thermofluctuation theory and the nonlinear parameter "b" (the change of modulus under loading). Substituting Eq. (10) into Eq. (5), and taking into account the values of kT, $\dot{\sigma}_0$, W_a, the formula for modulus of elasticity E, depending on stress, can be written as

$$E = E_0 - \frac{kT}{W_a}\frac{2\sigma}{\sigma_r}\ln\frac{\dot{\sigma}}{\dot{\sigma}_0} = E_0 - 808\frac{\sigma}{\sigma_r}(\ln\dot{\sigma} + 19.57)\ (\text{MPa}), \tag{11a}$$

Corresponding Eqs. (11b) and (11c) below are in the ratio form as a function of E_0 or E

$$\frac{E}{E_0} = 1 - \frac{kT}{W_a}\frac{2\sigma}{E_0\sigma_r}\ln\frac{\dot{\sigma}}{\dot{\sigma}_0} = 1 - 808\frac{\sigma}{E_0\sigma_r}(\ln\dot{\sigma} + 19.57) \tag{11b}$$

$$\frac{E}{E_0} = \frac{E}{E + 808\frac{\sigma}{\sigma_r}(\ln\dot{\sigma} + 19.57)} \tag{11c}$$

where σ, $\dot{\sigma}$ in MPa, MPa/s, respectively.

Therefore, Eq. (11a) confirms that the value of modulus of elasticity E depends on the temperature T, level of stress (σ/σ_r), and rate of loading ($\ln\dot{\sigma}$) [24,25]. It can be seen that the static modulus of elasticity at any level of loading has to be less than the initial modulus of elasticity (or dynamic modulus) E_0. Also the ratio of static modulus of elasticity to initial modulus of elasticity (or dynamic modulus) increases, when E_0 increases—Eq. (11b). The reciprocal of Eq. (11c), i.e. the ratio E_0/E of dynamic modulus to static modulus of elasticity, decreases as E increases (e.g., with age or strength). These conclusions correspond in general to existing experimental data for concrete [15,26,27]. Moreover, Eqs. (11a)–(11c) confirm the fundamental differences between static and dynamic modulus of elasticity.

Taking into account Eq. (10a), the Eq. (5) for strength can be reduced to the form

$$\sigma_r = \frac{kT}{W_a}\frac{2}{\frac{\partial E}{\partial \sigma}}\ln\frac{\dot{\sigma}}{\dot{\sigma}_0} = 808\frac{1}{\frac{\partial E}{\partial \sigma}}(\ln\dot{\sigma} + 19.57) \tag{11d}$$

The relation between strength and rate of loading is well known [28–31] but this fact acquires a physical explanation for concrete and other composite materials (graphite/epoxy, carbon/epoxy, glass–fiber reinforced, etc.) based on the thermofluctuation theory and nonlinear deformation (Eq. (11d)).

3. Comparison with experimental results and discussions

3.1. Relationship between moduli of elasticity

First of all it is important to correlate the nonlinear approach with commonly used relations.

As it was shown above, Eq. (6) between stress and strain is the equation of state that illustrates the behavior of a solid under loading. An equation similar to (6) describes the nonlinear response of concrete related to internal microcracking [32]. Portland Cement Association reported similar to parabola stress–strain curves obtained for cylindrical concrete specimens loaded in uniaxial compression [33].

Then from Eq. (6) it can be seen that at the maximum stress $(\partial\sigma/\partial\varepsilon) = 0$, and the maximum strain is $\varepsilon_{max} = E_0/2b$. Substituting ε_{max} in Eq. (6), we can find corresponding value of maximum stress

$$\sigma_{max} = \frac{(E_0)^2}{4b} \tag{6a}$$

Let assume that σ_{max} is the strength of concrete cylinder (f'_c), and E_0 is its modulus of elasticity (E). Then from Eq. (6a) follows the relation between elastic modulus and the strength in the form

$$E = (4bf'_c)^{1/2} = (4b)^{1/2}(f'_c)^{1/2} \tag{6b}$$

Eq. (6b) reflects the nonlinear behavior of concrete under loading, and provides the possible explanation of the relationship "modulus versus strength" for several ACI documents and many papers. Indeed, if the nonlinear parameter "b" is a constant in a wide range of concrete mixtures, and $(4b)^{1/2} = 4700$, then $E = 4700 * (f'_c)^{1/2}$ (MPa). Thus, from Eq. (6b), in particular case, follows the formula recommended by ACI 318 [34]. For corresponding range of concrete mixtures the average value of the nonlinear parameter equal: $b = (4700/2)^2 = 552.25 \times 10^4$ MPa.

For High-Strength Concrete ACI 363 [35] recommends another relation: $E = 3300(f'_c)^{1/2} + 6900$. For those range of concrete the nonlinear parameter is less than in ACI 318 and can be estimated as: $b \sim (3300/2)^2 = 272.25 \times 10^4$ MPa. Presented comparisons explain also the physical meaning of the constants in ACI documents [36].

Before comparing the modulus of elasticity at quasi-static and dynamic tests, it is a need to note here the following. The uniaxial compression loading leads to a total deformation of concrete, which includes elastic and inelastic part, whereas the dynamic tests cause little or no inelastic part of the entire deformation [13–18]. Simultaneously the range of strains for quasi-static method is usually (10^{-3}–10^{-6}); for resonant frequency method—(10^{-5}–10^{-6}); for ultrasound method—(10^{-7}–10^{-8}) [3]. Thus, in order to make a proper comparison between quasi-static and dynamic tests of the modulus of elasticity the values of strains (stresses) have to be specified. One can see also from Eqs. (11a)–(11c) that the modulus of elasticity is strain-rate sensitive. But this dependence is logarithmic, i.e. small in comparison to linear dependency. Therefore, the influence of the level of stress (strain) on the modulus of elasticity must be taken into consideration first.

The ratio of the quasi-static modulus of elasticity (E/E_0) and the ratio of the stress to the value of compressive strength (σ/σ_r) are shown in Fig. 1 for different values of E_0 assuming that the rate of loading is constant ($\dot{\sigma} = 0.24$ MPa/s). The values of the ratio E/E_0 were predicted according to Eq. (11b). One can see from Fig. 1, the ratio of the static modulus of elasticity (E/E_0) decreases when the ratio of compressive loading (σ/σ_r) increases. Because the strength of concrete usually increases as modulus of elasticity increases, Fig. 1 illustrates that for each ratio of stress the ratio of modulus decreases less as concrete's strength increases. For example, if the value of stress is within (0.3–0.4), and modulus of elasticity is between $(3–4) \times 10^4$ MPa, then the ratio E/E_0 decreases in the range (10–20)%. If the modulus of elasticity is approximately 2.1×10^4 MPa, then the ratio E/E_0 decreases in the range (20–30)%.

Table 1 shows the results, in the same manner as Fig. 1, but using dynamic tests for E_0 and compressive stress–strain measurements on cylinders 15 by 30 cm of concrete, ranging from rich to very lean mixes [13]. From the results for stresses less than σ_{min}, where the stress–strain curve coincides with dynamic (resonance) tests of Young's modulus E_0, follows. The range of corresponding strains is $(1–4) \times 10^{-5}$ (ε_{min}), the ratio ($\sigma_{min}/\sigma_r = \sigma_{min}/f'_c$), where f'_c is the strength, is small ($\sigma_{min}/f'_c = 0.01$–0.04), and the ratio E/E_0 is high: $E/E_0 = 0.98$–0.99 (Eq. (11b)). The stresses σ_{max} indicate the inelastic behavior of concrete. From Table 1 it can be seen that in accordance with experimental data the ratio E/E_0 (Eq. (11b)) of modulus of elasticity decreases ($E/E_0 = 0.89$–0.96) if the ratio of stress to compressive strength increases ($\sigma_{min}/f'_c = 0.10$–0.23).

It is of interest to note similar results for other applied static stress in the same order magnitude of loading. In Table 2 are shown the results of the deflection method using the bending tests for determining Young's modulus at two loads, $A = 46.2$ kg, and $B = 68.86$ kg, applied on 5 by 5 by 24 cm bars [13]. The data in Table 2 and statistical calculations confirm with reliability 100% that the average value of the Young's modulus decreases if the load increases. The difference between static modulus of elasticity E_A at load A and Young's modulus E_0 of the resonance method is statistically insignificant. But the corresponding difference at load B ($B > A$) is statistically significant. The average ratio $(E_A + E_B)/2E_0$ of the modulus of elasticity at quasi-static

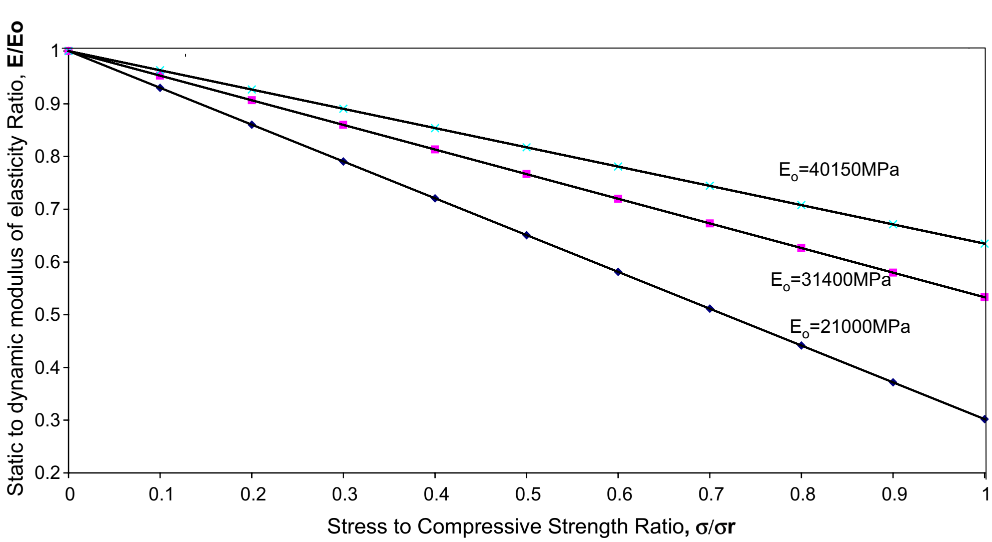

Fig. 1. Effect of stress to compressive strength ratio on the static to dynamic modulus of elasticity ratio.

Table 1
Experimental data [13] and the ratio of modulus of elasticity by Eq. (11b)

No.	Age	f'_c (MPa)	σ_{min} (MPa)	E_0 (MPa)	$\varepsilon_{min} = \sigma_{min}/E_0 \times 10^5$	σ_{min}/f'_c ratio	E/E_0, Eq. (11b)	σ_{max} (MPa)	σ_{max}/f'_c ratio	E/E_0, Eq. (11b)
1	28 days	53	1.38	36,544	3.77	0.03	0.99	6.90	0.13	0.95
2	28 days	47	0.69	36,544	1.89	0.01	0.99	6.90	0.15	0.94
3	28 days	53	1.38	39,302	3.51	0.03	0.99	5.52	0.10	0.96
4	28 days	46	1.38	37,923	3.64	0.03	0.99	6.90	0.15	0.94
5	28 days	34	1.38	35,165	3.92	0.04	0.98	6.90	0.20	0.92
6	28 days	24	0.69	31,717	2.17	0.03	0.99	5.52	0.23	0.89

Table 2
Static modulus at two different levels of loading, dynamic modulus [13] and results of statistical calculations

Series	Age, days	Deflection method		Resonance method, $E_0 \times 10^{-4}$ (MPa)	Ratio	
		A = 46.2 kg, $E_A \times 10^{-4}$ (MPa)	B = 68.86 kg, $E_B \times 10^{-4}$ (MPa)		E_A/E_0	E_B/E_0
J294	17	3.1	2.2	2.6	1.22	0.86
	17	2.6	2.3	2.6	1.00	0.92
	17	3.0	2.8	2.8	1.05	0.98
	17	2.6	2.6	2.8	0.93	0.90
	17	3.4		3.4	1.00	0.00
	17	3.6	3.1	3.4	1.04	0.90
	17	3.6	2.9	3.4	1.06	0.86
	17	3.4	3.0	3.4	1.02	0.88
233	13	2.8	2.3	2.8	1.00	0.85
	12	2.6	2.4	2.6	0.97	0.92
	11	2.6	2.4	2.6	0.97	0.92
	13	2.8	2.6	2.8	0.98	0.93
	12	3.0	2.5	2.8	1.05	0.88
	11	3.0	2.6	2.8	1.05	0.90
	13	3.0	2.8	3.0	1.00	0.93
	12	3.4	2.9	3.0	1.14	0.95
	11	2.8	2.9	3.0	0.91	0.95
Average		3.01	2.64	2.9	1.02	0.86
STDEV		0.37	0.27	0.3	0.07	0.22

loading and dynamic loading is less than one, which corresponds to Eq. (11b).

The same data were obtained from 320 pairs of comparative values of Young's modulus, namely: the secant modulus results at 50% of the ultimate flexural strength were slightly lower than those at 25% of ultimate [15]. From an exploratory study of fiber reinforced plastic bars it follows again that the difference between the elastic modulus for tensile static stress–strain method and the elastic modulus for impact resonance method is statistically significant, and that the static modulus is less than the dynamic modulus [37].

Eq. (11b) describes also qualitatively the general trend: at constant values of compressive loading and rate the ratio of modulus of elasticity becomes grater for high values of dynamic modulus and strength, which corresponds, for instance, to experimental data for concrete's cylinders at different ages [14].

In this connection attention should be paid to the data presented in Tables 3 and 4 for cylinders 15 by 30 cm of concrete made with five types of Long-Time Study (LTS) cements [16]. Tables 3 and 4 give the results for three cement contents (bold type): 334.1 kg/m^3, and 250.6 kg/m^3, 167.1 kg/m^3 (air-entrained), respectively. These tables provide the experimental values of the modulus of elasticity at 28 days, 1year, and 3 years as determined by the resonant method (E_0), compression methods (E), and static modulus derived from Eq. (11a). One can see for the three cement contents the total average value of error between predicted values of E (at $\sigma/\sigma_r = \sigma/f'_c = 0.4$, $\dot{\sigma} = 0.24$ MPa/s) and experimental values of E (the ratio σ/f'_c is unknown) is −17%, +8%, +4% at 28 days, 1 year, 3 year, respectively. It means that the inelastic behavior of concrete decreases as age increases, and that the nonlinear elastic model (6) as well as Eqs. (11a) and (11b) are promising for nondestructive methods for evaluating the static modulus of elasticity of old concrete.

In addition Table 5 contains data for concrete specimens tested at the age 1, 2, 7, 28 days. This table shows experimental results of static modulus (secant, chord), and dynamic (resonance) modulus

Table 3
Dynamic (E_0), static (E) modulus of elasticity for concretes made with LTS cements [16] and comparison with Eq. (11a)

Cement (kg/m^3)		$E_0 \times 10^{-4}$ (MPa)			$E \times 10^{-4}$ (MPa)			$E \times 10^{-4}$ (MPa): (11a)			Error (%)		
No.	type	28 days	1 year	3 years	28 days	1 year	3 years	28 days	1 year	3 years	28 days	1 year	3 years
334.1													
11	I	3.73	4.43	4.65	2.32	3.78	4.03	3.14	3.85	4.06	−35	−2	−1
11t	IT[a]	3.20	3.71	3.45	2.17	3.35	3.45	2.61	3.12	2.87	−21	7	17
15	I	3.65	4.33	4.65	2.77	3.70	4.04	3.07	3.74	4.06	−11	−1	−1
19A	I	3.44	4.10	4.49	2.43	3.79	3.94	2.85	3.52	3.90	−18	7	1
19B	I	3.59	4.12	4.55	2.39	3.83	3.89	3.00	3.53	3.96	−25	8	−2
19C	I	3.59	4.26	4.61	2.12	3.94	4.01	3.00	3.67	4.02	−41	7	0
Average		3.53	4.16	4.40	2.37	3.73	3.89	2.95	3.57	3.81	−24	4	2
21	II	3.54	4.34	4.56	2.61	4.16	4.10	2.96	3.76	3.97	−13	10	3
21T	IIT[a]	3.50	4.26	4.41	2.41	3.85	4.03	2.92	3.67	3.83	−21	4	5
31	III	3.39	4.14	4.21	2.61	3.64	3.65	2.80	3.56	3.63	−7	2	1
41	IV	3.43	4.25	4.59	2.68	3.91	4.02	2.85	3.66	4.00	−6	6	1
51	V	3.74	4.48	4.90	2.52	4.42	4.48	3.16	3.90	4.32	−25	12	4
Average		3.52	4.30	4.53	2.56	3.99	4.06	2.94	3.71	3.95	−15	7	3

[a] "Treated". Comparable to present day air-entraining cements.

Table 4
Dynamic (E_0), static (E) modulus of elasticity for concretes made with LTS cements [16] and comparison with Eq. (11a)

Cement (kg/m^3)		$E_0 \times 10^{-4}$ (MPa)			$E \times 10^{-4}$ (MPa)			$E \times 10^{-4}$ (MPa): (11a)			Error (%)		
No.	type	28 days	1 year	3 years	28 days	1 year	3 years	28 days	1 year	3 years	28 days	1 year	3 years
250.6													
11	I	3.51	4.07	4.26	2.25	3.65	3.77	2.92	3.48	3.67	−30	5	3
11T	IT[a]	3.32	3.45	3.59	2.02	3.03	3.04	2.73	2.86	3.00	−35	6	1
15	I	3.45	4.07	4.23	2.54	3.68	3.59	2.86	3.48	3.65	−13	5	−2
19A	I	3.19	3.93	4.25	2.16	3.77	3.85	2.60	3.34	3.66	−20	11	5
19B	I	3.29	3.92	4.19	2.18	3.73	3.75	2.70	3.34	3.60	−24	11	4
19C	I	3.40	4.08	4.23	2.34	3.81	3.61	2.81	3.50	3.65	−20	8	−1
Average		3.36	3.92	4.12	2.25	3.61	3.60	2.77	3.33	3.54	−23	8	2
21	II	3.21	4.19	4.45	2.38	3.99	4.00	2.62	3.61	3.86	−10	10	3
21T	IIT[a]	3.14	4.01	4.20	2.13	3.66	3.70	2.56	3.43	3.61	−20	6	2
31	III	3.34	3.87	4.02	2.54	3.64	3.45	2.75	3.28	3.43	−8	10	0
41	IV	3.08	3.95	4.30	2.21	3.72	3.70	2.50	3.36	3.71	−13	9	0
51	V	3.40	4.27	4.63	2.23	4.30	4.33	2.81	3.69	4.05	−26	14	7
Average		3.23	4.06	4.32	2.30	3.86	3.84	2.65	3.47	3.73	−15	10	3
167.1													
12	I	2.76	3.44	3.48	2.03	3.12	3.28	2.17	2.85	2.89	−7	8	12
23	II	2.71	3.41	3.59	1.97	3.08	3.32	2.12	2.82	3.01	−8	9	10
43A	IV	2.54	3.54	3.73	1.85	3.56	3.59	1.95	2.96	3.14	−5	17	12
Average		2.67	3.46	3.60	1.95	3.25	3.39	2.08	2.88	3.01	−7	12	11

[a] "Treated". Comparable to present day air-entraining cements.

[38] in comparison with corresponding prediction of Eq. (11a) at $\sigma/\sigma_r = \sigma/f'_c = 0.4$, $\dot{\sigma} = 0.24$ MPa/s. Each experimental result is an average of three measurements. The inherent error (−13%, bold type) is once again evident between the value of the secant modulus and the value of static modulus of Eq. (11a). This error is mainly due to the inelastic portion of the measured strain. The chord modulus was measured at lower strain level of stress–strain curve [38]. As a result the average error between chord modulus and predicted static modulus is 3% (Table 5, last column).

The presented data illustrate the influence of the value of applied stress on the value of the static modulus of elasticity. If the level of stress (strain) at static and dynamic loading is approximately in the same range then the difference between the static and dynamic modulus of elasticity becomes less, which also corresponds to the results in Table 1. Thus, Table 5 allows assuming that the nonlinear elastic model can be used for the nondestructive methods of evaluating the chord modulus of elasticity for early aged concrete.

Table 5
Secant (E_s), chord (E_{ch}), dynamic (E_0) moduli of elasticity [33] and comparison with Eq. (11a)

Age (days)	Modulus of elasticity (MPa)			E, Eq. (11a)	Error, %	
	Secant, E_s	Dynamic, E_0	Chord, E_{ch}		$(E_s - E)/E_s$	$(E_{ch} - E)/E_{ch}$
w/c = 0.4						
1	20,678	28,910	24,206	23,047	−11	5
3	23,912	34,006	27,636	28,143	−18	−2
7	25,872	36,260	31,066	30,397	−17	2
28	30,478	38,122	35,084	32,259	−6	8
w/c = 0.5						
1	16,856	24,402	18,816	18,539	−10	1
3	22,540	30,870	25,480	25,007	−11	2
7	24,304	33,810	28,322	27,947	−15	1
28	27,244	37,044	32,144	31,181	−14	3
Average					**−13**	**3**

Now let us compare Eq. (11c) with corresponding correlation (12) and (13) obtained from conventional static and dynamic (resonance) tests [25,39]

$$E = 1.25E_0 - 19\ (\mathrm{GPa}) \quad (12)$$

$$E/E_0 = 0.368 + 0.0871 \times 10^{-6}E\ (\mathrm{psi}) \quad (13)$$

or

$$E/E_0 = 1.25 - 23.75/(E \times 10^{-3} + 19)\ (\mathrm{MPa}) \quad (12a)$$

$$E/E_0 = 0.368 + 12.63 \times 10^{-6}E\ (\mathrm{MPa}) \quad (13a)$$

Eq. (12a) is derived from the general relation (12) given in BS 8110: Part 2:1985 for concrete at age of 28 days, and Eq. (13a) is from Eq. (13) for the static modulus of elasticity in the range $(3–5) \times 10^4$ MPa. Table 6 contains data for relations (12a) and (13a) and Eq. (11c) evaluated at constant ratio $\sigma/\sigma_r = \sigma/f_c' = 0.4$, $\dot{\sigma} = 0.24$ MPa/s. From Table 6 follows: the error between basic Eqs. (12a)–(13a), (12a)–(11c) ranges within −11% and +3%, but the range of error between experimental Eq. (13a) and theoretical Eq. (11c) is from −15% to +9%. It should be noted that the relationship (13) was found using 250 cylinders 15 by 30 cm cured under water for 90 days with a wide span of compressive strength ranging between (21–55) MPa. The static modulus was determined by the slope of the line joining the two points on the curve: $\sigma_{min} = 3.45$, and $\sigma_{max} =$ 13.79 MPa, i.e. the difference is equal $\sigma = 10.34$ MPa. Table 7 shows the experimental values of E/E_0, theoretical values of E/E_0 at $\dot{\sigma} = 0.24$ MPa/s for individual ratios $\sigma/\sigma_r = 10.34\ \mathrm{MPa}/f_c'$ at each value of strength and constant ratio $\sigma/f_c' = 0.4$. One can see for individual ratios the error between experimental Eq. (13a) and theoretical Eq. (11c) is in the range of −9% and +5%, which is less than the range of −13% and +11% for constant ratio of stress (Table 7).

Correlation between the static and the dynamic modulus for concrete mixes of limestone aggregates is also reported [40]

$$E = 1.033E_0 - 7.245\ (\mathrm{GPa}), \quad R^2 = 0.996 \quad (14)$$

$$E = 0.966E_0 - 4.914\ (\mathrm{GPa}), \quad R^2 = 0.981 \quad (15)$$

$$E = 1.005E_0 - 6.577\ (\mathrm{GPa}), \quad R^2 = 0.980 \quad (16)$$

R^2 is the coefficient of determination. Eq. (14) obtained for the mix of clean limestone aggregate without fly ash cured in water of 21 °C. Eq. (15) represents data for cylinders made from the same mix but cured in water of 35 °C and in the field. Eq. (16) accumulates data from all mixes (with and without fly ash, clean and dirty aggregates) cured under different conditions. One can see Eqs. (14)–(16) are similar to each other and theoretical Eq. (11a). It is important to note that the average value of the first term is equal one, and the average value

Table 6
The ratio static to dynamic modulus (E/E_0) for relations (12a), (13a) and (11c)

E (MPa)	E/E_0, Eq. (12a)	Eq. (13a)	Eq. (11c)	Error, %		
				(12a)–(13a)	(12a)–(11c)	(13a)–(11c)
27,800	0.74	0.72	0.82	3	−11	−15
31,000	0.78	0.76	0.84	2	−9	−11
34,500	0.81	0.80	0.85	0	−6	−6
38,000	0.83	0.85	0.87	−2	−4	−2
41,400	0.86	0.89	0.88	−4	−2	2
44,800	0.88	0.93	0.88	−6	−1	5
48,300	0.90	0.98	0.89	−9	1	9

Table 7
Comparison betweeen Eq. (13a) and Eq. (11c) at different ratio σ/f'_c

f'_c (MPa)	E (MPa)	E/E_0, Eq. (13a)	Ratio, $10.34/f'_c$	E/E_0, Eq. (11c)	Error (%), (13a)–(11c)	Ratio, $\sigma/f'_c = 0.4$	E/E_0, Eq. (11c)	Error (%) (13a)–(11c)
21	28,890	0.73	0.50	0.80	−9	0.4	0.83	−13
28	34,820	0.81	0.38	0.86	−7	0.4	0.86	−6
34	39,646	0.87	0.30	0.90	−4	0.4	0.87	0
41	43,714	0.92	0.25	0.92	0	0.4	0.88	4
48	47,162	0.96	0.21	0.94	3	0.4	0.89	6
55	50,127	1.00	0.19	0.95	5	0.4	0.90	11

of the second term equal 6244 MPa, which concedes with its counterpart in Eq. (11a) at $\sigma/\sigma_r = \sigma/f'_c = 0.42$, $\dot{\sigma} = 0.24$ MPa/s. The supporting data from the dynamic test results indicates that Eq. (11a) can be used to predict the static modulus of elasticity for aged concrete ranging between (1–28) days.

3.2. Relationship between strength and nonlinear parameter

From the original Eq. (6b) discussed in previous section follows that the strength of concrete is

$$f'_c = E^2/4b$$

This equation confirms a trend: the strength increases if the nonlinear parameter "b" decreases. Using 480 cylinders made of Types I, V, and V cement + fly ash concrete, it was found [41] that ACI 318 model accurately estimates concrete strength under 30 MPa, and ACI 363 over 30 MPa. It was shown in Section 3.1 also the corresponding value of nonlinear parameter: $b = 552.25 \times 10^4$ MPa; $b = 272.25 \times 10^4$ MPa, respectively. Hence, to find the right value of the strength for different mixtures of concrete there is a need to evaluate the value of nonlinear parameter.

To investigate the relationship between strength and nonlinear parameter $\frac{\partial E}{\partial \sigma}$ mix proportions of concrete having five water–cement ratios (0.35, 0.40, 0.45, 0.5, 0.55) were prepared. Young's Modulus, Poisson's ratio and compressive strength tests were performed on designated specimens according to ASTM C 469 and ASTM C 39 [42,43]. During the first cycle of loading the ratio of normal stress to corresponding strain, i.e. modulus of elasticity, was measured within strain range (10–1000)$\mu\varepsilon$. Then the nonlinear parameter dE/dσ as a slope between the modulus of elasticity and stress has been found from the data having a correlation coefficient at least −0.9. Table 8 shows the comparison of the experimental compressive strength results with corresponding predicted results using Eq. (11d) at $\dot{\sigma} = 0.24$ MPa/s [36]. Bold letters represent the average values of mechanical properties of concrete. It is seen in Table 8 that the average relative error of predicted strength does not exceed 10%. Table 9 shows that the averages values of the elas-

Table 8
Experimental and theoretical values of strength

w/c	E (MPa)	f'_c (MPa)	dE/dσ	f'_c (MPa) Eq. (11d)	Error (%)
0.5	31,000	40	338	43	−9
0.5	30,500	42	383	38	9
0.5	33,800	39	404	36	7
	31,800	**40**	**375**	**39**	**2**
0.45	35,800	57	258	57	0
0.45	35,500	40	314	47	−15
0.45	37,700	51	266	55	−7
	36,300	**49**	**279**	**53**	**−8**
0.4	34,800	52	321	46	12
0.4	32,000	53	221	66	−24
0.4	34,000	41	373	39	4
0.4	37,500	42	382	38	10
0.4	36,800	44	326	45	−2
	35,000	**47**	**325**	**47**	**0**
0.35	37,400	49	244	60	−23
0.35	40,800	58	256	57	1
0.35	37,500	48	250	59	−23
0.35	37,700	58	343	43	27
	38,300	**53**	**273**	**55**	**−5**

Table 9
Correlation "mechanical properties—w/c" of concrete, and the values of the nonlinear parameter "b"

w/c	E (MPa)	$dE/d\sigma$	f'_c (MPa)		$b = (E * dE/2d\sigma)\ 10^4$ from Eq. (7a) (MPa)
			Experiment	Eq. (11d)	
0.50	31,768	375	40	39	596.250
0.45	36,340	279	49	53	506.385
0.40	35,020	325	47	47	568.750
0.35	38,335	273	53	55	522.795
	Correlation coefficient, r				Average
	−0.86	0.71	−0.88	−0.75	$(548.545) \times 10^4$, MPa

tic modulus and strength increases as water–cement ratio decreases (correlation coefficient, r, equals −0.86, −0.88, respectively), but the averages values of the nonlinear parameter decreases as water–cement decreases (correlation coefficient, r, equals +0.71). The modulus of elasticity and the measured nonlinear parameter have opposing tendencies. In addition it is of interest to point out that the correlation coefficient between water–cement ratio and predicted strength of concrete is close to −0.75. Hence, the presented data indicate that modulus of elasticity and the measured nonlinear parameter is independent characteristics of concrete. It can be seen also that the difference between the average experimental value of the nonlinear parameter $b = 548.545 \times 10^4$ MPa (column 6) and calculated above from ACI 318 ($b = 552.250 \times 10^4$ MPa) is within 1%.

Therefore, from a practical point of view, it is important to find the nonlinear parameter of concrete in the field. Experimental evaluation of the nonlinear parameter can be made using dynamic tests based on the stress-wave propagation. For example, detailed observation on the effect of the magnitude of dynamic loading on the flexural resonant frequency showed that the fundamental frequency "always decreased with increase in the vibratory strain". For beams with dimensions 8 by 15 by 83 cm the relative resonance frequency shift was about −2% in the range (5–75) microstrain [44]. Similar data were obtained using the longitudinal resonance technique. For concrete prisms with dimensions 10 by 10 by 30 cm the relative resonance frequency shift was −0.5% in the range (0.1–7) microstrain [21,45]. Takings into account these results and the relationship between modulus of elasticity and resonance frequency, the nonlinear parameter $\frac{\partial E}{\partial \sigma}$ can be estimated as: $\frac{\partial E}{\partial \sigma} = \frac{\partial E}{E \partial \varepsilon} \sim 2\frac{\partial f}{f \partial \varepsilon}$ (f is the resonance frequency; ε is the amplitude of excitation). Thus, for beams and prisms, the nonlinear parameter was 571 and 1500, respectively.

Two nondestructive methods were developed based on the measurement of resonant frequency shift and phase shift [21,45]. The phase difference, ϕ, for two levels of excitation amplitude was found on several high performance concrete specimens with constant ultrasound pulse velocities respectively at age 3 days and 28 days. The obtained results proved that ultrasound pulse velocity (dynamic modulus of elasticity) and nonlinear parameter are independent because at each age the velocities were constant, but the nonlinear parameters were different for these specimens. It means that the nonlinear parameters are more sensitive to the structure of concrete than ultrasound velocities (linear parameters). Simultaneously it was shown that the more the phase shift, i.e. a nonlinear parameter, in specimen of high performance concrete the less its strength. A linear regression of cube strength R with phase shift ϕ produced $R = 66.5–6.84\phi$, with a mean-square error 10% in the investigated range satisfying the practical requirements.

4. Conclusions

The developed approach attempts to unify the methods for evaluating the relationship between static and dynamic modulus of elasticity as well as strength of concrete on the basis of its nonlinear deformation. Also from theoretical and experimental studies the following conclusions can be drawn.

1. The effect of nonlinear deformation of concrete on its moduli of elasticity may be analyzed using thermofluctuation theory of strength and the hypothesis that stress–strain dependency up to the peak strength can be described as a quadratic parabola.
2. The obtained theoretical relationship between static and dynamic modulus of elasticity indicates the following. The values of applied stress (strain) have to be specified first in order to make a proper comparison between static and dynamic moduli. The ratio of static modulus to dynamic modulus of elasticity (E/E_0) decreases as the ratio of loading increases. For each ratio of stress the ratio E/E_0 decreases less as concrete's initial (or dynamic) modulus increases. The influence of rate of loading is smaller than the influence of the level of stress.
3. Theoretical and experimental results for concrete made with 5 types of Long Time Study cements allows one to assume that Eqs. (11a)–(11c) may be useful for nondestructive methods of evaluating the static modulus of elasticity of old concrete.

4. Theoretical and experimental results for concrete made with 2 types of cement indicate that the nonlinear elastic model and the corresponding equations can be used for nondestructive methods for evaluating the chord modulus of elasticity for early aged concrete.
5. The general relation between the static and the dynamic modulus for concrete at the age of 28 days given in BS 8110: Part 2:1985 and the corresponding theoretical Eq. (11c) show that the latter is promising for evaluating the static modulus of elasticity using the value of the dynamic modulus of elasticity.
6. The similarity between the theoretical and the experimental relationships for concrete mixes of limestone aggregates provides the possibility to evaluate the static modulus from the results of the dynamic tests based on stress-wave propagation.
7. The presented data at static and dynamic loading indicate that the modulus of elasticity and nonlinear parameter is different and independent of concrete characteristics.
8. Nonlinear nondestructive methods can be used when conventional acoustic methods are not applicable for evaluating strength of concrete.
9. Although additional tests are needed for better statistical representation, the presented data illustrates the potential of the nonlinear approach, and indicates a new direction for evaluating the static modulus of elasticity and strength of concrete.

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