Pergamon

Cement and Concrete Research, Vol. 25, No. 2, pp. 408-416, 1995

0008-8846/95 $9.50+.00

0008-8846(95)00026-7

STATISTICS AND SIZE EFFECT IN CEMENTITIOUS MATERIALS

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(Refereed)
(Received April 1, 1994)

ABSTRACT The statistical size effect in cementitious beams is examined using the assumption that the critical threshold stress to cause a strain-softened fracture process zone to be initiated follows the Weibull distribution. The fictitious crack line model and the K-superposition theory, in combination with the statistical strain-softening relationship, are used to determine the strength of plain and notched beams. For notched beams the statistical effect on the strength is only important for small beams, but causes a large variation in all sizes of plain beams.

Introduction

The statistical theories of the fracture of brittle materials under predominantly tensile stresses are based on the weakest link concept originally proposed by Weibull(1,2). In this classic analysis, a structure is assumed to be composed of a large number of elements of varying strength. When one element fails, fracture is catastrophic and the structure fails. It has been subsequently shown that the Weibull theory can be interpreted in terms of flaws. Fracture, in this interpretation, occurs when the stress intensity factor at one of the flaws reaches the fracture toughness (3-6). The theory can be extended to account for the presence of a tougher second phase (7) or the existence of a crack growth resistance (8). Providing fracture is initiated at a single flaw, the size effect that is predicted by these theories is solely dependent of the distribution of the initial flaws. The mean strength of a small brittle specimen is greater than that of a large geometrically similar specimen simply because there is more chance of finding a large defect in the larger specimen. With cementitious materials stable microcracking occurs in a fracture process zone (FPZ) before catastrophic fracture and weakest link concepts cannot be applied directly. The microcracking in the FPZ in cementitious materials leads to strain softening in the FPZ which, in itself, causes a large size effect even if the material properties are deterministic (9-11).

The most well-known size effect law (SEL) is that of Bažant for geometrically similar specimens

$$\sigma_N = \frac{Bf_t}{\sqrt{1+\beta}} \tag{1}$$

where σ_N is the nominal fracture stress, f_t the tensile strength, $\beta = W/W_0$, W is the beam depth or other linear dimension, and B, W_0 are empirical constants (9,10). In effect the SEL is a first order linear elastic fracture mechanics solution obtained by using an equivalent crack whose length $a_e = a+\alpha a_p$, where a is the actual crack length, αa_p is a certain fraction, α, of the FPZ, a_p (10,11). In its original form this law has no statistical element, but Bažant and Xi (12) have considered the statistical variations in the strength of the cementitious material. They argue that there are two asymptotic limits to the SEL:

$$\begin{aligned} &\textit{for small } W/a_p, \quad \sigma_N \propto W^{-n/m} \\ &\textit{for large } W/a_p, \quad \sigma_N \propto W^{-1/2} \end{aligned} \tag{2}$$

where n is the number of spatial dimensions and m is the Weibull modulus of the strength variation and give a simple empirical equation

$$\sigma_N = \frac{Bf_t}{\sqrt{\beta^{2n/m}+\beta}} \tag{3}$$

that has these limits. Bazant and Xi (12) only consider notched beams of constant geometry (12).

Mazars et al. (13) have a different approach to size effect in cementitious materials. They use the concept of damage to analyze the fracture of cementitious materials. In this concept, the stiffness of an element decreases from its virgin value to zero as the damage, D, increases from zero to unity. They assume that no damage occurs until the strain reaches a threshold value. After the threshold has been reached, it is assumed that damage increases with non-local strain. With a deterministic threshold for strain this approach gives similar results to other non-local damage models. However, Mazars et al. (13) found that the strength of small beams was underestimated and a statistical threshold to damage, based on the Weibull distribution, gave better agreement. Mazars et al. also considered plain un-notched beams. For very large plain beams the slope of a log/log plot of strength against size has the Weibull predicted value of 1/m.

In this paper the effect of statistical variation in the strength of cementitious materials on the SEL is more fully explored by applying Weibull weakest link theory to the FPZ. Outside of the FPZ, microcracking is an extremely rare event and one that will cause practically no effect on the strength. The FPZ in laboratory sized specimens of cementitious materials has to be modelled in order to produce an accurate prediction of fracture. The two principal methods of deterministic modelling of the FPZ are the crack band (14-16) and the fictitious crack line (17-19). Both models are capable of accurately modelling the fracture of cementitious materials and the choice of method usually depends on which is simplest in the particular context. In the symmetrical specimens studied here, the fictitious crack line model is simplest and has been used throughout.

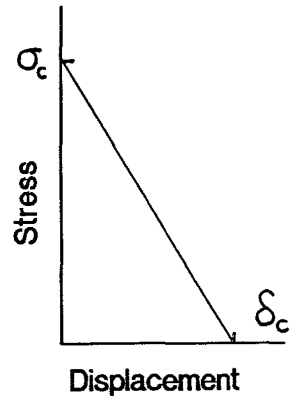

FIG. 1
Linear strain-softening relationship

Statistical Strain Softening

Outside of the FPZ it is assumed that the cementitious material is elastic and suffers no damage even in any region of high compressive stress. It is assumed that the FPZ is narrow and its width a material constant. The actual FPZ is not exactly constant, having more of a flame-like shape, but the non-local crack band model shows that its maximum width is not very size dependent (14). For the purpose of calculating the maximum load the FPZ is modelled by a fictitious extension to the true crack. A simple linear stress-displacement relationship is chosen for the strain softening behaviour of the FPZ (see Fig. 1). A bilinear model is more accurate in modelling the strain-softening behaviour (18), but is an unnecessary complication in this present study. As a FPZ grows so its chance of encountering extra weak zones will increase and the average stress carried will diminish. To avoid using a stochastic method, it is assumed that the stress threshold σ_c necessary to initiate strain-softening depends on the volume, V_p, of the FPZ, and is proportional to the Weibull mean strength so that

$$\frac{\sigma_c}{\sigma_o} = \left(\frac{V_{po}}{V_p}\right)^{1/m} \tag{4}$$

where σ_0 is the stress threshold for a standard FPZ of volume V_{p0}. The fracture energy of cementitious materials, G_f, is not a perfect material constant independent of size, but increases slightly with size to a plateau value (20-22). This increase in fracture energy with size must to some extent offset the decrease in σ_c. However, in this paper the size dependence of the fracture energy is neglected, and G_f is assumed to be a constant. Thus combined with the assumption that the stress-displacement relationship is linear, the critical crack tip opening displacement δ_c is given by

$$\delta_c = \frac{2G_f}{\sigma_c} \tag{5}$$

Modelling the fracture of beams statistically

The problem analyzed is a plain or centrally notched beam, with a span four times its depth W, loaded by a central load. The fictitious crack problem can be solved relatively simply by using standard stress intensity factors and the K-superposition theory the basis of which is that the total stress intensity factor, K, at the tip of the fictitious crack or FPZ is zero (23). There are two components to the total stress intensity factor: the stress intensity factor due to the applied load, K_a, and that due to the stress in the FPZ, K_p, which is negative. Thus

$$K = K_a + K_p = 0 \tag{6}$$

Since K_p depends on the crack opening displacement across the fictitious crack, it cannot be determined explicitly.

Strictly the stress in the FPZ depends upon the crack opening all along the faces of the fictitious crack. However, except in the initial stages of the development of the FPZ, the faces of the fictitious crack are nearly straight apart from a very small cusp right at the tip of the fictitious crack (23). The analysis is greatly simplified by assuming straight faces, since it is only necessary to iterate to find the crack tip opening displacement, δ_t. The error in the load induced by assuming that the fictitious crack faces remain straight is less than 5% (23).

Since the stress in the FPZ depends on its size, it is easiest to drive the program with the size of the FPZ, a_p. The values of the beam depth W, initial notch depth, a_0, and size of the FPZ, a_p, together with an initial guess of the crack tip opening displacement, δ_i, are used to calculate the load from the condition given by equation (6). The actual crack tip opening is then calculated for the assumed stress in the FPZ. An iterative routine then finds the actual value of the crack tip opening displacement for that particular size of FPZ and calculates the corresponding load. The size of the FPZ, a_p, is then increased to find the maximum load sustainable by the beam. In all cases maximum load occurs for $\delta_t < \delta_c$ though as the beam depth increases so δ_t approaches δ_c in the limit of an infinite beam.

The beam sizes have been non-dimensionalized by the characteristic length, l_{ch}, (17) defined by

$$l_{ch} = \frac{EG_f}{\sigma_0^2} \tag{7}$$

The results have been normalized on a beam whose non-dimensional depth, W, is unity. Thus V_{p0} is assumed to be the critical volume of the FPZ for a non-dimensional beam whose depth is unity.

Size effect results

Bažant and Xi (12) suggest that an appropriate value of the Weibull modulus, m, is 12. Thus the Weibull modulus for cementitious materials is probably bounded by 10 and 20. Three different classes of beams have been analyzed:

(i) Beams with deterministic strain-softening characteristics ($m = \infty$).
(ii) Beams of constant width, m = 20, 10.
(iii) Beams whose width is proportional to their depth, m = 20, 10.

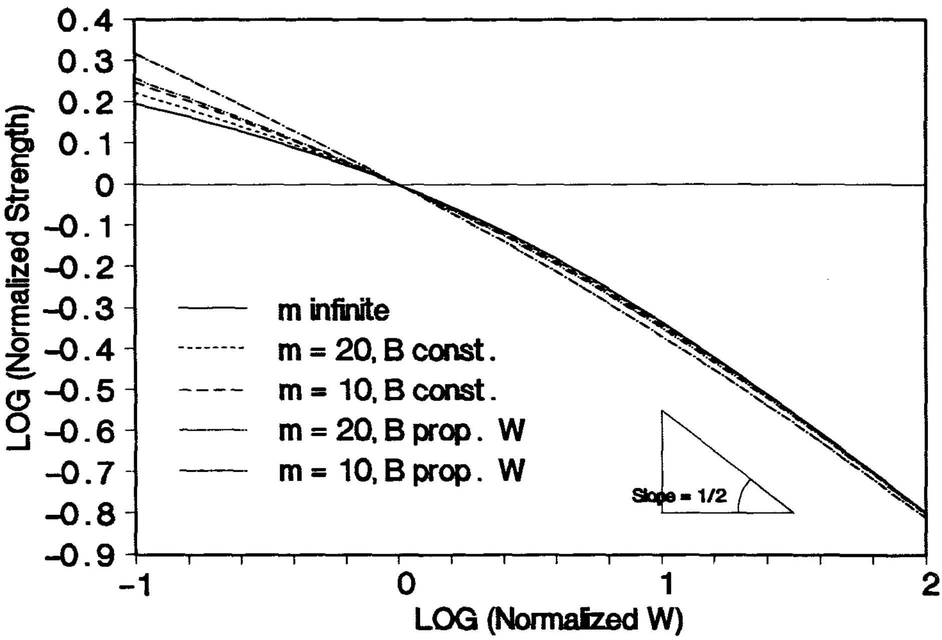

FIG. 2
Normalized strength of notched beams $a_0/W = 0.3$

Two sets of beams have been analyzed. In the first beam the relative notch depth, a_0/W is 0.3 and a logarithmic plot of the normalized strength against the normalized beam depth is shown in Fig. 2. There is little statistical effect in large notched specimens because the FPZ is almost fully developed at final fracture and it is small compared with the notch size. Under these conditions it is the fracture energy, G_f, that controls the fracture not the stress. In the limit for an infinite sized beam, the fracture occurs when

$$K_a = (EG_f)^{1/2} \tag{8}$$

and the slope of the logarithmic plot tends to -1/2 as the beam size increases, as predicted by Bažant's size effect law (9,10,12). For small specimens fracture occurs before the FPZ has developed very much and the FPZ is comparable or even larger than the notch size. Under these conditions the actual strain softening relationship becomes important. Since physically the size of the FPZ must decrease as the specimens get smaller, the strength of small specimens is statistically greater than indicated from a deterministic model. Obviously if the width of the specimens are in proportion to their depth then the increase in strength is greater than if the width of the specimens is kept constant, because the decrease in FPZ volume is greater.

Similar results are given by the approximate relationship (equation 3) of Bažant and Xi (12) and these results, normalized so that, for deterministic strain-softening, the normalized strength is unity for a beam whose normalized depth is unity, and equal to that predicted by the present analysis for a beam, whose normalized depth is 100, are presented in Fig. 3. However, n, the number of spatial variables has been interpreted somewhat differently to Bažant and Xi (12). They take n=2, if the width of the beam is constant, and n=3 for complete geometrical similarity, whereas in Fig. 3, n=1 and 2 respectively which we believe to be more in keeping with the concept of a FPZ whose width is approximately constant, independent of the size of the specimen. Obviously there is little difference between the strengths given in the two graphs for W>1, because of the normalization procedure. However, it is seen that the present results predict less statistical effect for the smaller beams. The difference is caused because in the model of Bažant and Xi (12) the FPZ in small specimens is assumed to be proportional in size to the beam, whereas in the present results the size of the FPZ decreases more slowly with decrease in beam size.

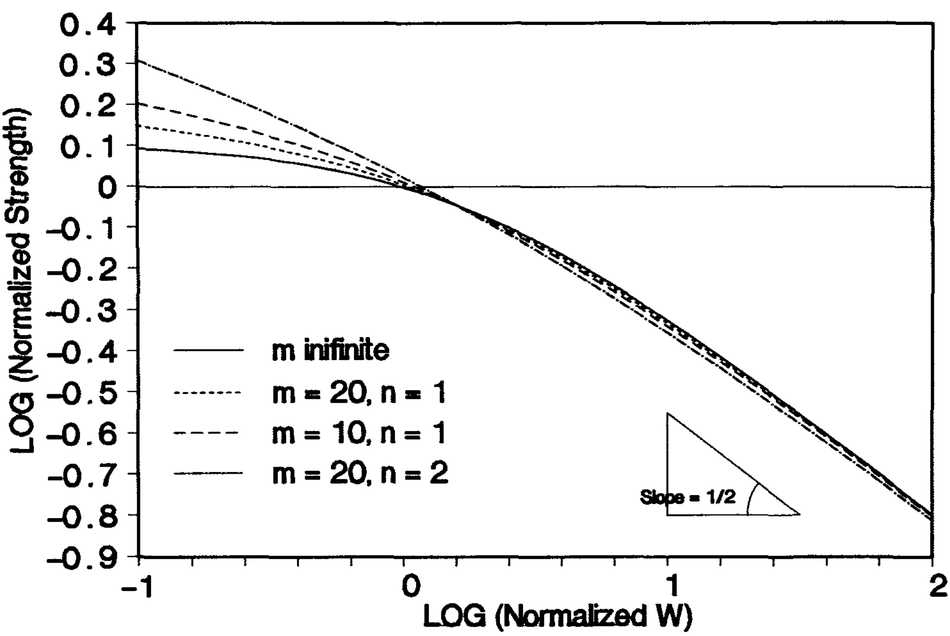

FIG. 3
Approximate normalized strength of notched beams $a_0/W = 0.3$ (12)

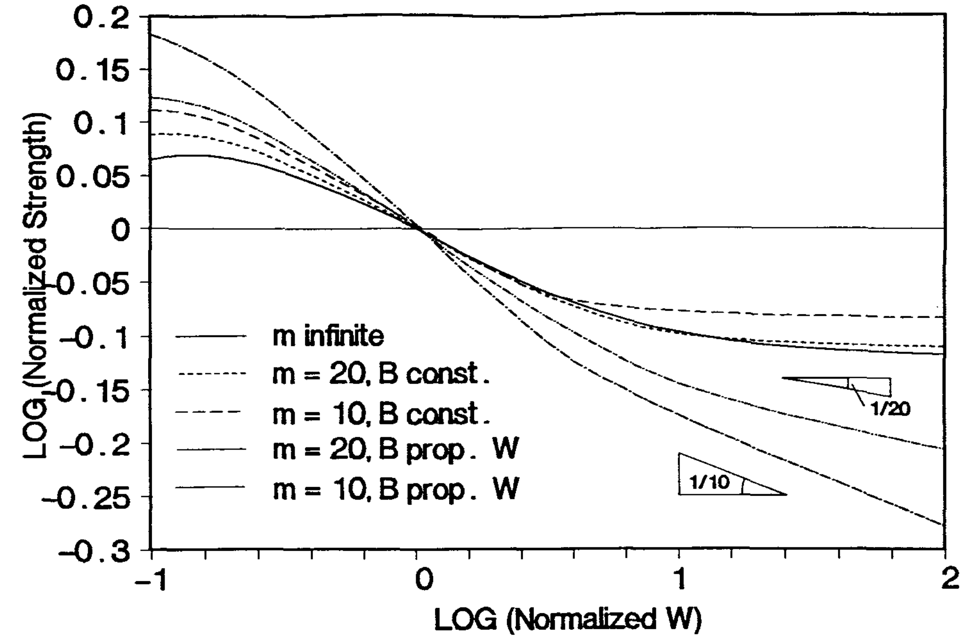

FIG. 4
Normalized strength of "plain" beams

The second set of beams analyzed is representative of plain, un-notched, beams. The fictitious crack model can be used to determine the size effect if the strain-softening relationship is deterministic (11). However, because of a fundamental difference in the behaviour of notched and plain beams, this model cannot be used directly if the strain-softening relationship follows Weibull's law. In a notched beam a FPZ starts to form immediately a load is applied to the beam, but in a plain beam, a FPZ does not initiate until the maximum elastic stress attains the critical stress, σ_c. If this stress is given by equation (4), the critical stress is infinite at the initiation of a FPZ and the strength of the beam correspondingly infinite. In practice the inhomogeneity, inherent in cementitious materials, will lead to an early initiation of a FPZ. This early stage in the formation of a FPZ is not important except in very large beams, because final fracture only occurs after the FPZ has grown to a relatively large size. To avoid the artifact of the present model, results have been obtained for beams that have a small defect or crack whose absolute size is the same for all beams. The non-dimensional defect size chosen is 10^{-2}, so that the smallest beam (W=0.1) has a defect whose relative size, $a_0/W=10^{-1}$, and the largest beam (W=100) has a defect with $a/W=10^{-4}$. A logarithmic plot of the normalized strength of such beams is shown in Fig. 4. The beams whose strain-softening characteristics are deterministic tend to a non-dimensional strength of unity as they get larger. The slight dip in strength of these deterministic beams for W<0.16 is caused by the assumed defect, for a perfect beam the strength would continue to increase. The effect on the mean strength of the statistical strain-softening relationship is much more marked in these "plain" specimens. The strength of the small specimens is increased by the statistical variation, as was the case for the notched beams shown in Fig. 2. The size of the FPZ for "plain" beams reaches its most developed state for near unit sized beams and then decreases to an asymptotic value as the beams increase in size. If the width of the beams is constant, the volume of the FPZ decreases causing the statistical mean strength of the large beams to be larger than the deterministic ones. However, length of the FPZ decreases less slowly than the depth of the beam increases. Thus if the width of the beam is kept in proportion to the depth of the beam, the FPZ increases in volume with size and the statistical strength is less than the deterministic value. For large beams the length of the FPZ is constant and the volume therefore in proportion to the depth of the beam. Thus, for beams whose width is in proportion to their depth, the slope of the logarithmic plot of strength against beam size tends to -1/m. This latter asymptotic trend was noted by Mazars et al.(13).

Conclusions

The fictitious crack line model can be used to analyze the effect of statistical variations in the strain-softening behaviour of the FPZ, and the K-field superposition method provides a simple computational tool to enable the size effect to be determined.

The size effect for notched beams is little affected by statistical variations in the strain-softening relationship except for very small size beams. However, the statistical influence on the size effect in plain, un-notched beams is much more marked. Interestingly the statistical effect on the strain-softening relationship causes an increase in the mean strength if the width of the beams is kept constant, but decrease if the width of the beams are in proportion to their depth.

References

1. W., Weibull, Ingen. Veterskaps Akad. Hand., No 151, 1939.

2. W., Weibull, J. Appl. Mech., 18, 293 (1951).

3. A.De S. Jayatilaka, and K. Trustram, J. Mater. Sci., 12, 1426 (1977).

4. R.A. Hunt, and L.N. McCartney Int. J. Fract., 15 365 (1979).

5. L.N. McCartney Int. J. Fract., 15, 477 (1979).

6. K. Trustrum, and A.De S. Jayatilaka, J. Mater. Sci., 18, 2765 (1983).

7. X.-Z. Hu, B. Cotterell, and Y.-W. Mai, Proc. R. Soc. (Lond.), A 401, 251 (1985).

8. K. Kendall, N.McN. Alford, S.R. Tan, and J.D. Birchall, J. Mater. Res., 1, 120 (1986).

9. Z.P. Bažant, J.-K. Kim, and P.A. Pfeiffer, J. Struct. Eng. ASCE, 112 289 (1986).

10. Z.P. Bažant, and M.T.Kazemi, Int. J. Fract. 44, 111 (1990).

11. B. Cotterell, and Y.-W. Mai, Fracture of Engineering Materials and Structures, ed: S.H. Teoh, and K.H. Lee, Elsevier Applied Science, London, 348 (1991).

12. Z.P. Bažant, and Y. Xi, J. Eng. Mech. ASCE, 117 2623 (1991).

13. J. Mazars, G. Pijaudier-Cabot, and C. Saodouridis, Int. J. Fract. 51 159 (1991).

14. Z.P. Bažant, and B.-F. Lin, J. Struct. Eng. ASCE, 114, 2493 (1989).

15. Z.P. Bažant, and J. Ozbolt, J. Eng. Mech. ASCE, 116, 248 (1991).

16. Z.P. Bažant, Macroscopic Behaviour of Heterogeneous Materials from Microstructure, (Proceedings of Winter Annual Meeting of the ASME, Anaheim, November) ASME, New York, 153 (1992).

17. A. Hillerborg, M. Modéer, and P.E. Petersson, Cem. Concr. Res. 6, 773 (1976).

18. F.H. Wittmann, P.E. Roelfstra, H. Mihashi, Y.-Y. Huang, X,-Y. Zhang, and N. Nomura, Mater. & Struct. 20, 103 (1987).

19. A. Hillerborg, Int. J. Fract., 51, 95 (1991).

20. A. Hilsdorf, and W. Bramshuber, Applications of Fracture Mechanics to Cementitious Composites, ed: S.P. Shah, Northwestern University, 255 (1984).

21. A. Hillerborg, Mater. & Struct., 18, 291 (1985).

22. X.-Z. Hu, and F.H. Wittmann, Mater. & Struct., 25, (1992).

23. B. Cotterell, P. Paramasivam, and K.Y. Lam, Mater. & Struct., 25, 14 (1992).