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DOUBLE INCLUSION MODEL FOR APPROXIMATE ELASTIC MODULI OF CONCRETE MATERIAL

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ABSTRACT

The overall elastic moduli of concrete composite materials are investigated in this study by employing the theory of micromechanics. A method based on Mori-Tanaka Theory and double inclusion can provide a more accurate evaluation of the average elastic relationships of concrete composite materials. The inclusions are divided into two groups: fine aggregate and coarse aggregate. Moreover, the overall elastic moduli of the concrete composite materials are given as a function of properties and volume fraction of the following three components: fine aggregate, coarse aggregate, and cement paste. A comparison is also made between the theoretical results and published experimental data.

Introduction

By considering concrete as a two-phase material, Aitcin and Mehta (1), Baalbaki et al. (2) demonstrated that the elastic modulus of concrete is influenced by the elastic properties and volume fraction of aggregates. Hirsch (3) derived an equation to express the elastic modulus of concrete in terms of an empirical constant, and also provided some experimental results for the elastic modulus of concrete with different aggregates.

The overall mechanical behavior of composite materials has been extensively studied. Voigt's (4) approximation yielded the upper bound and the Reuss's (5) approximation yielded the lower bound of the average elastic moduli. Hashin and Shtrikman (6) proposed the variational principle to find bounds on the average elastic moduli of composite materials which were better than the Voigt and Reuss bounds. Hansen (7) developed mathematical models to predict the elastic moduli of composite materials based on the individual elastic modulus and vole fraction of the components.

Mori and Tanaka (8) applied the concept of average field to analyze macroscopic properties of composite materials. The average field in a body contains inclusions with eigenstrain. In addition, the shape effect of dispersoids was introduced in Eshelby's (9) method to assess the properties of composite materials. The recent development of evaluating overall elastic modulus and overall elastic-plastic behavior was reviewed by Mura (10), Nemat-Nasser and Hori (11).

A new approach of average elastic relationships of concrete with two different spherical

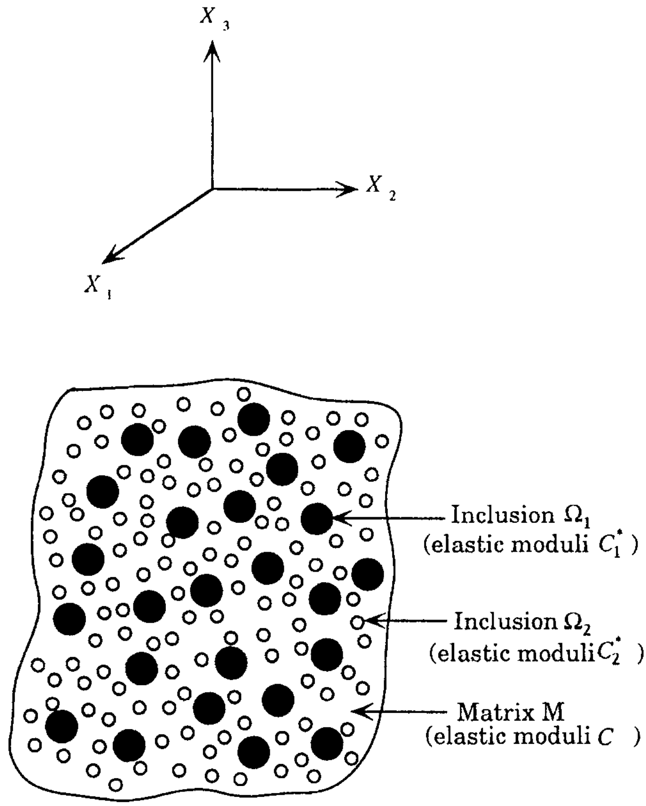

FIG. 1.
Two kinds of sphere inclusions embedded in an isotropic body.

inhomogeneities is evaluated in this study by employing Mori-Tanaka Theory and Eshelby's Method. A comparison is also made between the results of theoretical predictions and published experimental data.

Homogenization Model

A sufficiently large body D is considered to contain two types of inclusions. The inclusions are with the same shape and distributed randomly (Fig. 1). Consider fine aggregate as one type of inclusions, $\Omega_1 = \sum_{i=1}^{N} \Omega_{1i}$, with elastic modula $\underset{\sim}{C_1^*}$ and volume fraction f_1, and coarse aggregate as the other type of inclusions, $\Omega_2 = \sum_{i=1}^{N} \Omega_{2i}$, with elastic moduli $\underset{\sim}{C_2^*}$ and volume fraction f_2. The domain surrounding the inhomogeneities is referred to as the matrix (cement paste) M which has an elastic moduli $\underset{\sim}{C}$ (Fig. 1). As a result, the average elastic moduli of the composite materials (concrete) $\underset{\sim}{\bar{C}}$, can be evaluated.

Consider an infinite composite material subjected to an applied uniform stress $\underset{\sim}{\sigma}^0$ at infinity. In the so-called equivalent inclusions method (12), a composite material is simulated by a homogeneous material with uniform stiffness $\underset{\sim}{C}$ and distributing eigenstrains $\underset{\sim}{\varepsilon_1^*}$ in Ω_1 and $\underset{\sim}{\varepsilon_2^*}$ in Ω_2. The disturbances of the stress caused by the inhomogeneities in the concrete i.e., deviations from $\underset{\sim}{\sigma}^0$ are denoted by $\underset{\sim}{\sigma}^{\Omega_1}$, $\underset{\sim}{\sigma}^{\Omega_2}$, and $\underset{\sim}{\sigma}^M$ for two inclusions and matrix,

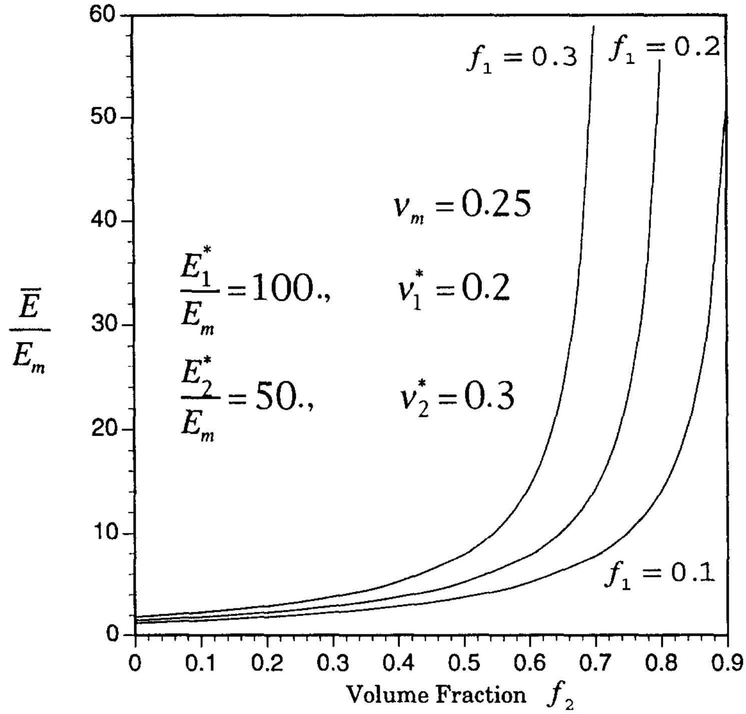

FIG. 2.
Young's modulus vs. the volume fraction of sphere inhomogeneities.

respectively. Mura (12) proposed that the strain deviation ($\Delta\tilde{\gamma}$) caused by the eigenstrain in the inclusion can be obtained as follows

$$\Delta\gamma_{ij}(x) = -\int_{\Omega_1} C_{klmn}\,\varepsilon^*_{mn}(x')\,\frac{1}{2}\{G_{ik,lj}(x-x') + G_{jk,li}(x-x')\}dx. \tag{1}$$

The solution for the equivalent inclusion problem which simulates the actual elastic state in the concrete is obtained by applying Hooke's law and employing the average stress quantity in Ω_1 and Ω_2,

$$\begin{aligned}\tilde{\sigma}^0 + \langle\tilde{\sigma}^{\Omega_1}\rangle &= \tilde{C}\left\{\tilde{C}^{-1}\left(\tilde{\sigma}^0 + \langle\tilde{\sigma}^M\rangle\right) + \langle\Delta\tilde{\gamma}_1\rangle - \langle\tilde{\varepsilon}^*_1\rangle\right\}\\ &= \tilde{C}^*_1\left\{\tilde{C}^{-1}\left(\tilde{\sigma}^0 + \langle\tilde{\sigma}^M\rangle\right) + \langle\Delta\tilde{\gamma}_1\rangle\right\},\end{aligned} \tag{2a}$$

$$\begin{aligned}\tilde{\sigma}^0 + \langle\tilde{\sigma}^{\Omega_2}\rangle &= \tilde{C}\left\{\tilde{C}^{-1}\left(\tilde{\sigma}^0 + \langle\tilde{\sigma}^M\rangle\right) + \langle\Delta\tilde{\gamma}_2\rangle - \langle\tilde{\varepsilon}^*_2\rangle\right\}\\ &= \tilde{C}^*_2\left\{\tilde{C}^{-1}\left(\tilde{\sigma}^0 + \langle\tilde{\sigma}^M\rangle\right) + \langle\Delta\tilde{\gamma}_2\rangle\right\},\end{aligned} \tag{2b}$$

where "$\langle\ \rangle$" is the average over Ω_1 and Ω_2. $\tilde{\varepsilon}^*_1$ and $\tilde{\varepsilon}^*_2$ are eigenstrains in Ω_1 and Ω_2, and $\Delta\tilde{\gamma}_1$ and $\Delta\tilde{\gamma}_2$ are the strain deviations caused by $\tilde{\varepsilon}^*_1$ and $\tilde{\varepsilon}^*_2$ in Ω_1 and Ω_2. The average strain

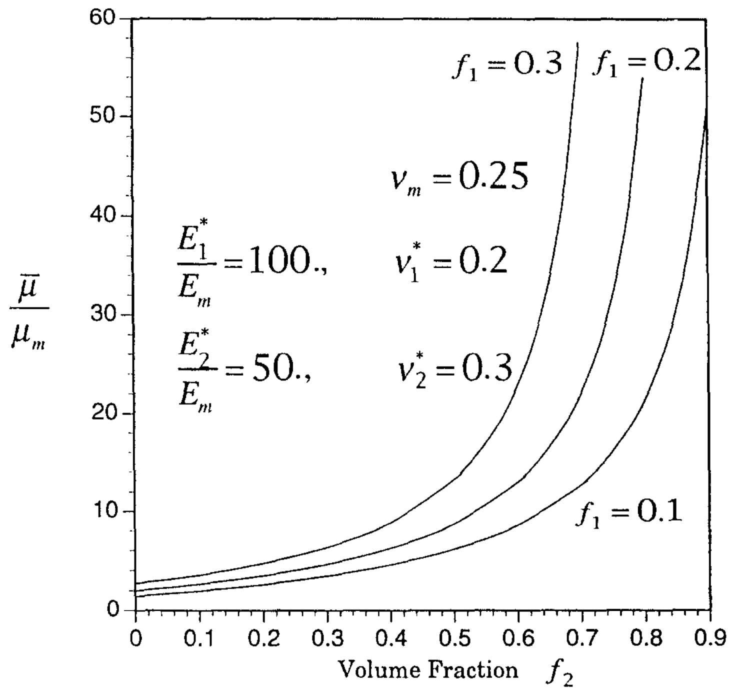

FIG. 3.
Shear modulus vs. the volume fraction of sphere inhomogeneities.

deviation for the inclusions can be expressed by

$$\langle \Delta \underset{\sim}{\gamma}_1 \rangle = \underset{\sim}{S}_1 \langle \underset{\sim}{\varepsilon}_1^* \rangle, \tag{3a}$$

$$\langle \Delta \underset{\sim}{\gamma}_2 \rangle = \underset{\sim}{S}_2 \langle \underset{\sim}{\varepsilon}_2^* \rangle, \tag{3b}$$

where $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ are the Eshelby tensor for a sphere inclusion when it isolately exists in an infinite homogeneous medium.

Since the average of the stress disturbance $\underset{\sim}{\sigma}$ must be zero, i.e.,

$$\int_D \sigma_{ij}\, dx = 0. \tag{4}$$

With simple substitution, eqn (4) can be rewritten as

$$f_1 \langle \underset{\sim}{\sigma}^{\Omega_1} \rangle + f_2 \langle \underset{\sim}{\sigma}^{\Omega_2} \rangle + (1 - f_1 - f_2) \langle \underset{\sim}{\sigma}^{M} \rangle = 0, \tag{5}$$

where f_1 and f_2 are the volume fraction of fine aggregates and coarse aggregates. Substituting eqns (3a) and (3b) into eqns (2a) and (2b) yields

TABLE 1
The Volume Fraction of Sand and the Young's Modulus of Mortar

Anson and Newman (1966)

A_1	A_2	A_3
Phase 1 (Cement Paste) $[E_m = 12\ GPa]$	Phase 2 (Sand) $[E_s = 80\ GPa]$	Composite (Mortar)
	Volume Fraction	$E_M(GPa)$
	0.322	21.90
	0.417	26.00
	0.487	28.00
	0.544	28.30
	0.589	30.40

$$\langle \underset{\sim}{\sigma}^{\Omega_1} \rangle = \langle \underset{\sim}{\sigma}^{M} \rangle + \underset{\sim}{C}\,(\underset{\sim}{S}_1 - \underset{\sim}{I}) \langle \underset{\sim}{\varepsilon}_1^{*} \rangle, \tag{6a}$$

$$\langle \underset{\sim}{\sigma}^{\Omega_2} \rangle = \langle \underset{\sim}{\sigma}^{M} \rangle + \underset{\sim}{C}\,(\underset{\sim}{S}_2 - \underset{\sim}{I}) \langle \underset{\sim}{\varepsilon}_2^{*} \rangle, \tag{6b}$$

where $\underset{\sim}{I}$ is the unit tensor. From eqns (5), (6a), and (6b), it follows that

$$\langle \underset{\sim}{\sigma}^{M} \rangle = -\left[f_1 \underset{\sim}{C}\,(\underset{\sim}{S}_1 - \underset{\sim}{I}) \langle \underset{\sim}{\varepsilon}_1^{*} \rangle + f_2 \underset{\sim}{C}\,(\underset{\sim}{S}_2 - \underset{\sim}{I}) \langle \underset{\sim}{\varepsilon}_2^{*} \rangle \right], \tag{7}$$

$$\langle \underset{\sim}{\sigma}^{\Omega_1} \rangle = (1 - f_1) \underset{\sim}{C}\,(\underset{\sim}{S}_1 - \underset{\sim}{I}) \langle \underset{\sim}{\varepsilon}_1^{*} \rangle - f_2 \underset{\sim}{C}\,(\underset{\sim}{S}_2 - \underset{\sim}{I}) \langle \underset{\sim}{\varepsilon}_2^{*} \rangle, \tag{8a}$$

$$\langle \underset{\sim}{\sigma}^{\Omega_2} \rangle = -\, f_1 \underset{\sim}{C}\,(\underset{\sim}{S}_1 - \underset{\sim}{I}) \langle \underset{\sim}{\varepsilon}_1^{*} \rangle + (1 - f_2) \underset{\sim}{C}\,(\underset{\sim}{S}_2 - \underset{\sim}{I}) \langle \underset{\sim}{\varepsilon}_2^{*} \rangle. \tag{8b}$$

By substituting eqns (7), (8a), and (8b) into eqns (2a) and (2b), then solving eqns (2a) and (2b) for $\langle \underset{\sim}{\varepsilon}_1^{*} \rangle$ and $\langle \underset{\sim}{\varepsilon}_2^{*} \rangle$ yields

$$\langle \underset{\sim}{\varepsilon}_1^{*} \rangle = \alpha\, \underset{\sim}{\sigma}^{0} \tag{9a}$$

$$\langle \underset{\sim}{\varepsilon}_2^{*} \rangle = \beta\, \underset{\sim}{\sigma}^{0} \tag{9b}$$

α and β are shown in the Appendix.

Equations (2a) and (2b) indicate that the average elastic strains $\langle \underset{\sim}{\gamma}^{\Omega_1} \rangle$ and $\langle \underset{\sim}{\gamma}^{\Omega_2} \rangle$ are given by

$$\langle \underset{\sim}{\gamma}^{\Omega_1} \rangle = \underset{\sim}{C}^{-1} \langle \underset{\sim}{\sigma}^{M} \rangle + \langle \Delta\, \underset{\sim}{\gamma}_1 \rangle, \tag{10a}$$

$$\langle \underset{\sim}{\gamma}^{\Omega_2} \rangle = \underset{\sim}{C}^{-1} \langle \underset{\sim}{\sigma}^{M} \rangle + \langle \Delta\, \underset{\sim}{\gamma}_2 \rangle. \tag{10b}$$

The average elastic strain $\langle \underset{\sim}{\gamma}^{M} \rangle$ of the disturbance in matrix M is given as

TABLE 2
The Volume Fraction of Gravel and the Young's Modulus of Concrete

Anson and Newman (1966)

B_1	B_2	B_3
Phase 1 (Mortar) [E_M = 28.30 GPa]	Phase 2 (Gravel) [E_g = 69 GPa]	Composite (Concrete) E_C(GPa)
	Volume fraction f	Young's Modulus
	0.18	34.90
	0.25	34.20
	0.28	35.40
	0.30	36.20
	0.35	38.60
	0.40	39.60

$$\langle \gamma^M \rangle = C^{-1} \langle \sigma^M \rangle. \tag{11}$$

From eqns (10a), (10b), and (11), the total average strain $\langle \bar{\gamma} \rangle$ of the body is given as

$$\begin{aligned} \langle \bar{\gamma} \rangle &= \gamma^0 + f_1 \langle \gamma^{\Omega_1} \rangle + f_2 \langle \gamma^{\Omega_2} \rangle + (1 - f_1 - f_2) \langle \gamma^M \rangle \\ &= (C^{-1} + f_1 \alpha + f_2 \beta) \sigma^0 \\ &= \bar{C}^{-1} \sigma^0, \end{aligned} \tag{12}$$

where γ^0 is the applied strain. $\langle \gamma^{\Omega_1} \rangle$, and $\langle \gamma^{\Omega_2} \rangle$ are the average elastic strain in Ω_1 and Ω_2. The average elastic compliance $\bar{C}^{-1}$ is evaluated from eqn (12). Therefore, the average elastic moduli tensor of concrete composite materials is given by

$$\bar{C} = (C^{-1} + f_1 \alpha + f_2 \beta)^{-1}. \tag{13}$$

Results and Comparison of Theory with Experimental Data

As shown in Fig. 1, two types of spherical inclusions are embedded in an isotropic infinite matrix. For elasticity calculations, the Young's moduli E^*_1 and E^*_2 of the first and second type inhomogeneities are 100 and 50 times the matrix E_m, respectively. The Poisson ratios of the matrix, and two inhomogeneities are 0.25, 0.2, and 0.3, respectively. The Eshelby's tensors S_1 and S_2 for sphere are obtained according to previous literature (10). By calculating C, C_1^*, and C_2^*, and substituting them into eqn (13), the average elastic moduli of the composite materials $\bar{C}$

TABLE 3
The Volume Fraction of (Sand+Gravel) and the Young's Modulus of Concrete
Anson and Newman (1966)

C_1	C_2	C_3
Phase 1 (Paste)	Phase 2 (Sand+Gravel)	Composite (Concrete) $E_C(GPa)$
	Volume fraction f	Young's Modulus
	0.63	34.90
	0.66	34.20
	0.67	35.40
	0.68	36.20
	0.70	38.60
	0.73	39.60

can be obtained. From $\bar{C}$, the Young's modulus $\bar{E}$ and shear modulus $\bar{\mu}$ of the composite materials can be calculated. The average dimensionless parameters of $\bar{E}/E_m$ and $\bar{\mu}/\mu_m$ are shown in Fig. 2 and Fig. 3 as functions of f_2 (the volume fraction of the second type of inhomogeneities) with various f_1 (the volume fraction of the first type of inhomogeneities).

Anson and Newman (13) provided experimental results for the Young's modulus of mortar and concrete with different aggregate volume fractions. The experimental data for cement paste and mortar with W/C = 0.5 are selected to compare with the theoretical results. Considering concrete as a two-phase composite materials, two groups of experimental results are accumulated: (a) cement paste-sand, and (b) mortar-gravel group. In the first group of experimental results, cement paste is considered as phase 1 and sand is considered as phase 2. Table 1 lists the Young's modulus of mortar with the volume fraction ranging from 0.32 to 0.589. In the second set of experimental results, mortar is considered as phase 1 which originates from the first set of experimental results, (Table 1); in addition, gravel is considered as phase 2. Table 2 displays the Young's modulus of concrete with the volume fraction ranging from 0.18 to 0.40. Table 3 shows the Young's modulus of concrete which contains cement paste in phase 1 and contains sand-gravel in phase 2. The results in Table 2 and Table 3 are obtained from the same experiment.

The theoretical model studied in this paper considers concrete to be a three phase composite with spherical inclusions. The Young's modulus of cement paste in phase 1 is $E_m = 12$ GPa, sand in phase 2 is $E_s = 80$ GPa, and gravel in phase 3 is $E_g = 69$ GPa. The Poisson ratio of cement paste v_m, sand v_s, and gravel v_g is assumed as 0.22, 0.21, and 0.23, respectively. The volume fractions of sand are calculated from Tables 2 and 3, and the volume fractions of gravel are the same as those found in Table 2. The overall elastic moduli of the concrete are calculated from eqn (13) as a function of Young's modulus, Poisson ratio, and volume fraction of the following three components: cement paste, sand, and gravel. Table 4 displays the average theoretical Young's moduli and experimental results of concrete. Evidently, the theoretical results correlate fairly with the experimental data.

TABLE 4
The Calculated Results and Experimental Data for Young's Modulus of Concrete

Cement Paste (Phase 1) E_m = 12 GPa ν_m = 0.22	Sand (Phase 2) E_s = 80 GPa ν_s = 0.21	Gravel (Phase 3) E_g = 69 GPa ν_g = 0.23	Concrete $\bar{E}$ (GPa) (Calculated)	Concrete* E_c(GPa) (Measured)	$\frac{\bar{E} - E_c}{E_c}$ (%)
f_0	f_1 (C_2-B_2)	f_2 (B_2)			
0.374	0.446	0.18	32.14	34.90	-7.9
0.342	0.408	0.25	33.92	34.20	-0.8
0.328	0.392	0.28	34.73	35.40	-1.9
0.319	0.381	0.30	35.29	36.20	-2.5
0.297	0.353	0.35	36.74	38.60	-4.8
0.274	0.326	0.40	38.28	39.60	-3.3

*Anson and Newman (1966)

Conclusions

The elastic modulus of concrete is influenced by the elastic properties and the volume fraction of the matrix (cement paste), fine aggregate, and coarse aggregate. A theoretical model for estimating the elastic moduli of concrete composite materials is proposed to consider concrete as a three phase composite. The theoretical predictions correlate sufficiently with previous experimental results.

Nilsen and Monteiro (14), Simeonov and Ahmad (15) studied the transition zone of concrete, which is a very interesting topic. Introducing one kind of inclusion to simulate the transition zone in concrete is also a challenging model for the elastic behavior of concrete.

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Appendix

The calculation of parameters α and β

$$A = \left[(1 - f_1)\underset{\sim}{C} + f_1 \underset{\sim}{C}_1^{*}\right]\left(\underset{\sim}{S}_1 - \underset{\sim}{I}\right) - \underset{\sim}{C}_1^{*}\underset{\sim}{S}_1$$

$$B = \left[(1 - f_2)\underset{\sim}{C} + f_2 \underset{\sim}{C}_2^{*}\right]\left(\underset{\sim}{S}_2 - \underset{\sim}{I}\right) - \underset{\sim}{C}_2^{*}\underset{\sim}{S}_2$$

$$M = \left(\underset{\sim}{C}_1^{*} - \underset{\sim}{C}\right)\left(\underset{\sim}{S}_2 - \underset{\sim}{I}\right)$$

$$N = \left(\underset{\sim}{C}_2^{*} - \underset{\sim}{C}\right)\left(\underset{\sim}{S}_1 - \underset{\sim}{I}\right)$$

$$\langle \underset{\sim}{\varepsilon}_1^{*} \rangle = -\left(\underset{\sim}{I} - f_1 f_2 A^{-1} MB^{-1} N\right)^{-1} A^{-1}\left[f_2 MB^{-1}\left(\underset{\sim}{C}_2^{*}\underset{\sim}{C} - \underset{\sim}{I}\right) + \left(\underset{\sim}{C}_1^{*}\underset{\sim}{C} - \underset{\sim}{I}\right)\right]\underset{\sim}{\sigma}^{0}$$

$$= \alpha\, \underset{\sim}{\sigma}^{0}$$

$$\langle \underset{\sim}{\varepsilon}_2^{*} \rangle = B^{-1}\Big\{f_1 N\left(\underset{\sim}{I} - f_1 f_2 A^{-1} MB^{-1} N\right)^{-1} A^{-1}\left[f_2 MB^{-1}\left(\underset{\sim}{C}_2^{*}\underset{\sim}{C} - \underset{\sim}{I}\right) - \left(\underset{\sim}{C}_1^{*}\underset{\sim}{C} - \underset{\sim}{I}\right)\right]$$

$$+ \left(\underset{\sim}{C}_2^{*}\underset{\sim}{C} - \underset{\sim}{I}\right)\Big\}\underset{\sim}{\sigma}^{0}$$

$$= \beta\, \underset{\sim}{\sigma}^{0}$$

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