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GENERIC FORM OF STRESS-STRAIN EQUATIONS FOR CONCRETE

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ABSTRACT

Advances in concrete technology over the years have made it possible to produce concrete of varying strengths, ranging from low to high, to meet the various demands of the concrete construction industry. The constitution of bulk concrete material is different for different grades of concrete. Their constitutive responses to applied loading are invariably different, and hence, it is anticipated that a single mathematical equation is not sufficient to represent the expected wide range of constitutive stress-strain behaviour of the concrete materials. This paper reviews the generic form of equations for the constitutive stress-strain relationship of different grades of concrete at monotonic uniaxial compression. These equations are shown to evolve from the general stress-strain equation developed by the author (21). An experimental investigation has been conducted to verify the use of these generic forms of equations in representing the ascending branch of the stress-strain relationship of various grades of concrete. *© 1998 Elsevier Science Ltd*

PAPERS

Introduction

Concrete is a composite material composed of aggregates of varying sizes embedded in a matrix of hardened cement paste. Its responses to external load depends on a complex combination of many factors. When a prismatic concrete specimen having an aspect ratio of about 2 to 3 is loaded in monotonic uniaxial compression at a constant stress or strain rate (1,2), its typical stress-strain behaviour is approximately elastic up to about 30% of the peak stress, $\sigma_{c,u}$, measured in N/mm^2, of the specimen. Thereafter, the stress-strain curve shows a gradual increase in curvature up to about 0.9 $\sigma_{c,u}$, whereupon it bends sharply to become almost horizontal at the peak stress. Under a carefully controlled strain rate, a descending branch of the stress-strain curve beyond the peak stress can be obtained (3,4).

Numerous mathematical equations for the non-linear constitutive stress-strain relationship of concrete at uniaxial compression have been developed (5–10). Generally, these equations have utilised some empirical relationships to represent the suspected nature of the deformations of concrete at increasing load in order to derive mathematical functions to fit the experimental data. The known fact that deformations of concrete are attributed to the process of progressive microcracking within the bulk concrete material (11–13) has led to further development of analytical models for predicting the short-term, rate-independent, stress-strain behaviour of concrete under complex loading regimes involving multi-axial loading and load reversals (14–20). Most of these models are phenomenological in nature and invariably contain a large number of state variables and parameters to cover all possible responses of the concrete material to the defined loading regimes.

Concrete is a highly complex composite material and its constitutive response to loading is closely dependent on its composition as well as its internal microscopic structure. It is anticipated that a single mathematical equation is not sufficient to represent the expected wide range of constitutive behaviour for different grades of concrete material. This paper investigates the generic forms of equations for the ascending branch of the constitutive stress-strain relationship of different grades of concrete under monotonic uniaxial compression. These generic equations are shown to evolve from the general stress-strain equation derived by the author (21).

Generic Stress-Strain Equation for Concrete at Uniaxial Compression

Equation 1 is the short-term, rate-independent, general stress-strain equation for a prismatic concrete specimen, having an aspect ratio of 2.5, under monotonic uniaxial compression (21).

$$\frac{\sigma_c}{\sigma_{c,u}} = \frac{\varepsilon_c}{\varepsilon_{c,u}} e^{\left[1-\frac{\varepsilon_c}{\varepsilon_{c,u}}\right]} \tag{1}$$

where σ_c is the compressive stress measured in N/mm^2, ε_c is the compressive strain, and $\varepsilon_{c,u}$ is the compressive strain at $\sigma_{c,u}$.

Equation 1 can be rewritten in the following form through the use of the exponential power series expansion:

$$\sigma_c = \frac{2.7182\,(E_{c,u})\,(\varepsilon_c)}{1 + \left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right) + \frac{1}{2}\left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right)^2 + \frac{1}{6}\left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right)^3} \tag{2}$$

Terms having the fourth power and above in the denominator of Eq. 2 have been omitted. To investigate the relative usefulness of Eq. 1 or 2 in representing the constitutive stress-strain responses of concrete at monotonic uniaxial compression, the respective empirical equations proposed by Desayi and Krishnan (6), Saenz (22), Tulin and Gerstle (23), Popovics (7), Carreira and Chu (8), and Tsai (9), for the stress-strain curve of standard laboratory cylindrical concrete specimens at monotonic uniaxial compression are reproduced below for comparison. For easy reference to Eq. 1 or 2, the Author's notations for equivalent terminologies are used in the various equations by others.

(i) Equation proposed by Desayi and Krishnan (6):

$$\sigma_c = \frac{2\,(E_{c,u})\,(\varepsilon_c)}{1 + \left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right)^2} \tag{3}$$

Equation 3 had also been proposed by Todeschini et al. (24).

(ii) Equation proposed by Saenz (22):

$$\sigma_c = \frac{(E_o)\,(\varepsilon_c)}{1 + \left(\frac{E_o}{E_{c,u}} - 2\right)\left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right) + \left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right)^2} \tag{4}$$

where E_o is the initial tangent modulus at $\varepsilon_c = 0$, measured in kN/mm^2. When $E_o = 2E_{c,u}$ (twice the secant modulus at $\sigma_{c,u}$, measured in kN/mm^2), Eq. 4 reverts to Eq. 3.

(iii) Equation proposed by Tulin and Gerstle (23):

$$\sigma_c = \frac{3\,(E_{c,u})\,(\varepsilon_c)}{2 + \left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right)^3} \tag{5}$$

(iv) Equation proposed by Popovics (7):

$$\sigma_c = \frac{n\,(E_{c,u})\,(\varepsilon_c)}{n - 1 + \left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right)^n} \tag{6}$$

The n power in Eq. 6 is obtained from $E_o/E_{c,u} = \{n/(n-1)\}$. Equation 6 is exactly the same as that proposed by Carreira and Chu (8) except that the symbol, β, is used in place of n in the equation. β has been defined as “a material parameter that depends on the shape of the stress-strain diagram,” especially the descending branch. When $n = 2$, Eq. 6 reverts to Eq. 3, and when $n = 3$, it reverts to Eq. 5.

(v) Equation proposed by Tsai (9):

$$\frac{\sigma_c}{\sigma_{c,u}} = \frac{M\left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right)}{1 + \left(M - \frac{N}{N-1}\right)\left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right) + \left(\frac{1}{N-1}\right)\left(\frac{\varepsilon_c}{\varepsilon_{c,u}}\right)^N} \tag{7}$$

In Eq. 7, M is given by the ratio $E_o/E_{c,u}$, and N, expressed as a function of $\sigma_{c,u}$, is defined as “a factor to control the steepness rate for the descending portions of the stress-strain relation” of concrete at monotonic uniaxial compression (9). When $M = 2$ and $N = 2$, Eq. 7 reverts to Eq. 3.

Comparison of Eq. 2 with Eqs. 3 to 7 shows clearly that the latter equations are generic forms of Eq. 2. The necessary experimental parameters to plot Eq. 1 or 2 are $E_{c,u}$ and $\varepsilon_{c,u}$, or equivalently, $\sigma_{c,u}$ and $\varepsilon_{c,u}$, respectively. An arbitrary set of $\sigma_{c,u}$ and $\varepsilon_{c,u}$ data is used in Eqs. 1, 3, and 5 to plot the respective curves as shown in Figure 1. Equations 6 and 7 are not plotted because they require the prior prerequisite determination of the various constants from the experimental data for substitution into the respective equations. As shown in Figure 1, at any individual strain value less than $\varepsilon_{c,u}$, both Eqs. 3 and 5 yield stress values lower than that given by Eq. 1, i.e., the respective ascending branch of Eqs. 3 and 5 is lower than that of Eq. 1.

Stress-Strain Curve for High Strength Concrete

Advances in concrete technology have now made it possible to produce high strength concrete having cube strengths exceeding 100 N/mm^2. The stress-strain response of high

TABLE 1
Mix proportion and characteristic strength of concrete.

Mix S/No.	Ratio by weight A/C	Ratio by weight W/C	Water (kg)	Cement (kg)	Sand (kg)	Aggregate (kg)	28-day f_{cu} (N/mm^2)	Prism strength $\sigma_{c,u}$ (N/mm^2)
1	4	0.4	180	450	765	1035	48.9	45.5
2	4	0.5	215.5	431	733	991	39.1	35.1
3	4	0.6	248	413	702	950	29.4	27.4
4	5	0.4	154	385	818	1107	47.0	40.5
5	5	0.5	185.5	371	788	1067	34.2	31.7
6	5	0.6	215	358	761	1029	27.2	26.9
7	6	0.4	134.5	336	857	1159	40.7	37.8
8	6	0.5	162.5	325	829	1121	30.8	30.5
9	6	0.6	189	315	803	1087	26.7	23.5

$f_{c,u}$, 28-day cube strength.

strength concrete under uniaxial short-term loads is approximately linear up to a higher percentage of the maximum stress and the strain corresponding to maximum stress increases with strength (25). It has been suggested that the ascending branch of Eq. 6 describes, fairly accurately, the stress-strain behaviour of high strength concrete (25,26).

Experimental Programme

An experimental programme had been conducted to verify the use of Eqs. 1, 3, and 5 in representing the ascending branch of the stress-strain curves of various grades of concrete under uniaxial compression. A total of nine mixes had been designed for three water-cement and three aggregate-cement ratios. The ratio of fine to coarse aggregate was 2:3. Table 1 shows the mix proportions per cubic meter of fresh concrete for each of the nine mixes. Ordinary Portland cement, washed sand, and 20 mm nominally maximum size crushed granite aggregate were used throughout. For each mix, three 100-mm cubes and five 100 $\times$ 100 $\times$ 500 mm prisms were cast. All cast specimens were demoulded the day after casting and thereafter continuously moist-cured for the next 27 days inside a fog room. The cubes were tested to determine the average standard 28-day cube strength (27) in accordance to BS 1881: Part 116. Each prism was sawn into half to yield a pair of 100 $\times$ 100 $\times$ 250 mm test prisms. The ends of each test prism are ground plane and perpendicular to its sides by using a proprietary end preparation machine grinder. Three prepared prisms were tested to obtain the average prism strength. On each of three other prisms, a 30-mm gauge length electrical resistance strain gauge was attached in parallel to the long side at the center of each of the four sides of an individual test prism. The remaining four test prisms were set aside for cyclic loading tests reported elsewhere (28). Owing to the necessary time required to prepare each prism for testing, as well as the time constraints in scheduling the testing of individual specimen, all the test prisms from each mix were invariably tested at an age of about 2 months from the date of casting. A 60-ton universal testing machine, operated in the compression mode, was used to test each prism at a loading rate of 0.075 $N/mm^2/s$. The load and the corresponding strain readings were recorded by a multi-channel data logger and stored in a computer.

Experimental Results and Discussion

Ascending Branch of the Stress-Strain Relationship of Concrete

The nine concrete mixes of Table 1 yielded 28-day cube strengths that ranged between 25 to 50 N/mm^2. These strengths represent the typical range of concrete grades commonly used in the local construction industry. The average ultimate strength of three test prisms at the age of 2 months for each mix are also given in Table 1. The stress-strain data for a typical prism of each of the nine mixes and the theoretical curves of Eqn. 1, 3, and 5 are plotted as shown in Figures 2 to 10, respectively. For each mix, the set of data chosen to plot the respective stress-strain data in Figures 2 to 10 is one in which an individual average strain reading is within 5% to 10% difference from the largest value of the corresponding readings from the four strain gauges. Comparison of experimental results with the theoretical curves for other prisms of the nine mixes are given in (28).

The stress-strain data of prisms from Mix 3, 5, 6, 8, and 9 fit the theoretical curve of Eq. 1 very well, whilst those of prisms from Mix 2 and 7 fit closer to the theoretical curve of Eq. 3, and those of prisms from Mix 1 and 4 fit closer to the theoretical curve of Eq. 5. These results show a consistent trend of behaviour for various grades of concrete. The theoretical curve of Eq. 1 fits concrete grades of approximately 30 N/mm^2 or below, whilst that of Eq. 2 fits concrete grades around the region of 35 to 40 N/mm^2, and that of Eq. 5 fits concrete grades of approximately 45 N/mm^2 and above. This trend of results effectively confirms the observation (25,26) that the ascending branch of Eq. 5 or, essentially that of Eq. 6 for any value of n greater than 3, describes fairly accurately the stress-strain behaviour of stronger concrete. The improved overall homogeneity within the bulk concrete material and better stiffness characteristics of stronger concrete effectively result in lesser difference between the initial tangent modulus, E_o, and the secant modulus at peak load, $E_{c,u}$. From Eq. 1, it is noted that $E_o = 2.72E_{c,u}$, whereas from Eq. 3, $E_o = 2E_{c,u}$, and from Eq. 5, $E_o = 1.5E_{c,u}$. From Eq. 6, for n=4, 5, and 6, $E_o = 1.33E_{c,u}$, $1.25E_{c,u}$, and $1.2E_{c,u}$, respectively. Hence, as the value of n increases, the difference between E_o and $E_{c,u}$ decreases, resulting in the rising curve becoming more linear. Thus, the ascending portion of the curve of Eq. 6 is a better fit to the experimental stress-strain data points of stronger concrete.

Descending Branch of the Stress-Strain Curve

The curve of Eq. 1 also yields a descending branch beyond the peak stress point. There are many attempts to derive empirical equations for the descending branch of the stress-strain curve (4,6–10,23,25). Invariably, many of these equations require the prior determination of experimental constants or parameters for use in conjunction with the respective equations to fit the post-peak experimental data points. For low strength concrete, the post-peak data points for the descending branch of the stress-strain curve can be obtained by controlled strain rate on the test specimen after the peak stress point. For example, when the straining rate is carefully regulated near to and beyond the peak stress point to allow the bulk concrete material to loose its remaining rigidity gradually, the logging of data points in the post-peak region is possible (4). The ultimate failure of stronger concrete can be explosive, resulting in multiple fragmentation of the bulk concrete material, thus making the logging of post-peak stress-strain data very difficult. Even when stringently controlled testing technique is imposed on the test specimen to prevent the type of instantaneous failure, the few post-peak data points that are logged result in very fast decay of the descending branch of the stress-strain curve (29).

It is noted from Figure 1 that beyond the peak stress point, the descending branch of Eq. 1 lies between those of Eqs. 3 and 5, respectively, with that of Eq. 3 being the highest of the three branches. It has been reported (25) that the descending branch of Eq. 6 does not decay fast enough after the peak stress point to represent the post-peak responses of high strength concrete. It is then suggested that the addition of a factor *k* in the denominator term of Eq. 6 will increase the post-peak decay (25). The resulting stress-strain relationship for high strength concrete is:

$$\sigma_c = \frac{n\, E_{c,u}\, \varepsilon_c}{n - 1 + \left(\dfrac{\varepsilon_c}{\varepsilon_{c,u}}\right)^{nk}} \tag{8}$$

When ε_c is less than $\varepsilon_{c,u}$, k is equal to 1, for which Eq. 8 essentially is the same as Eq. 6. When ε_c is greater than $\varepsilon_{c,u}$, k is greater than 1 . Expressions for k and n have been suggested (30) as follows:

$$k = 0.67 + \frac{\sigma_{c,u}}{62} \tag{9}$$

$$n = 0.8 + \frac{\sigma_{c,u}}{17} \tag{10}$$

Thus, it is possible to vary the generic forms of Eq. 8 to produce different descending branches of the stress-strain curve to suit various prescribed testing conditions in the post-peak test regime.

Conclusions

Generic equations can be derived from the generalised form of the short-term, rate-independent, stress-strain relationship for concrete at uniaxial compression to describe the corresponding constitutive responses of different grades of concrete. The ascending branch of the curve of Eq. 1 fits the experimental stress-strain data of low strength concrete, whilst that of Eq. 3 fits the experimental stress-strain data of medium strength concrete, and that of Eq. 5 or 6 fits the experimental stress-strain data of high strength concrete. Generic forms of the descending branch of the stress-strain curve of concrete can also be derived to fit the post-peak data of different grades of concrete.

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