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Stress analysis of expansive reactions in concrete

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Abstract

The expansion of reinforced concrete, which may be caused by either physical or chemical attack, may compromise the life span and integrity of a structural member. Presented below is a 2-D stress analysis of the expansion (or contraction) experienced by an inclusion (aggregate, reinforced bar, or air void) embedded in the cement paste matrix. The stresses and strain fields are determined using the complex representation of Muskhelishvili. The model permits that the reaction products have different elastic moduli than the original inclusion and it allows for expansion (or contraction) along the interface. Stresses caused by corrosion of reinforced concrete and by ice formation in concrete are discussed.

Keywords: Corrosion; Degradation; Expansion; Freezing and thawing; Long-term performance

1. Introduction

Deleterious reactions in reinforced concrete may cause volumetric changes in one or more of its phases, thereby generating complex stress fields. The cement paste matrix is particularly sensitive to tensile stresses, which often cause cracking. In plain concrete, expansion due to chemical attack may occur close to an aggregate (i.e., during the alkali–aggregate reaction) or in the matrix (i.e., during a delayed ettringite formation). Volumetric changes may also occur as a result of physical attack, such as during ice formation and salt precipitation. Volume changes in reinforced concrete may be caused by corrosion of the reinforcing bar, which generates corrosion products with a larger volume than the original material.

Expansive reactions are complex, and because of the heterogeneity existing in concrete, they develop in distinct regions. Therefore, to estimate the magnitude of the stresses caused by the expansive reactions, some simplified geometrical assumptions become necessary. Although the restrictions inherent in applying simplified geometry can be easily overcome through the use of numerical models, there is much to learn from the use of closed-form solutions where the interplay of the relevant variables becomes clear.

Goltermann [1] did a comprehensive study of the stresses developed during chemical reactions; Garboczi [2] extended this work by introducing a layer around a spherical inclusion. This paper introduces a model where reaction products with distinct elastic moduli are allowed to expand or shrink. The model is then applied to stresses that occur (a) during corrosion of reinforced concrete and (b) during ice formation in plain concrete.

2. Stress analysis

This paper considers a detailed 2-D model and uses the complex solution representation of Muskhelishvili [3] to derive the displacements and stresses for the model. The 2-D model is well suited for the analysis of a long steel bar where the cross-section is modeled as a circular inclusion in the 2-D domain. For a realistic model of spherical voids and inclusions in concrete, a 3-D analysis must be performed. This can be done using the 3-D complex function solution developed by Piltner [4]. Here, the details of the derivation of the displacement functions are explained only for the 2-D model, whereas a comparison of the radial displacement for the 3-D case with the 2-D case is included.

Fig. 1 shows the physical model used to predict the stresses generated during expansive reactions in concrete, where an inclusion (aggregate, reinforced bar, or air void) of radius r_i is surrounded by reaction products. The boundary

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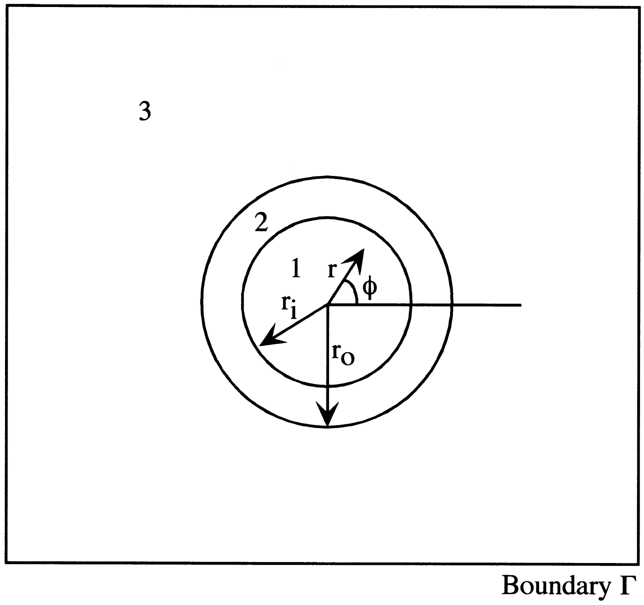

Fig. 1. Definition of the three domains with different material properties.

between reaction products and concrete lies on $r = r_o$. The material parameters of the three areas defined in Fig. 1 are denoted by μ_1, μ_2, μ_3 and κ_1, κ_2, κ_3, as they appear in Muskhelishvili [3]. The parameters are related by

$$\mu_i = \frac{E_i}{2(1+\nu_i)} \quad \text{and} \quad \kappa_i = \begin{cases} \frac{3-\nu_i}{1+\nu_i} & \text{for plane stress} \\ 3-4\nu_i & \text{for plane strain} \end{cases}$$

where μ is the shear modulus, E is the Young's modulus and ν is the Poisson's ratio.

For most cases, a state of plane strain is assumed. Using the complex representation of Muskhelishvili [3], the radial displacements u_r and the circumferential displacements u_θ for the three domains [Eqs. (1–3)] can be written in terms of complex functions $\phi_j(z)$, $\Psi_j(z)$:

$$2\mu_1(u_r + iu_\theta) = e^{-i\theta}\left[\kappa_1\phi_1 - z\overline{\phi'_1} - \overline{\Psi_1}\right] \tag{1}$$

$$2\mu_2(u_r + iu_\theta) = e^{-i\theta}\left[\kappa_2\phi_2 - z\overline{\phi'_2} - \overline{\Psi_2}\right] \tag{2}$$

$$2\mu_3(u_r + iu_0) = e^{-i\theta}\left[\kappa_3\phi_3 - z\overline{\phi'_3} - \overline{\Psi_3}\right] \tag{3}$$

where $z = x + iy = re^{i\theta}$ and $j = 1, 2, 3$.

The advantage of using this specific model is that the equilibrium equations are automatically satisfied with any choice of complex functions. We only have to choose appropriate complex valued functions for our problem and compute the free coefficients to satisfy the boundary conditions. The associated stresses are given by Eqs. (4,5):

$$\sigma_{rr} - i\tau_{r\theta} = \phi'_j + \overline{\phi'_j} - [\bar{z}\phi''_j + \Psi'_j]e^{2i\theta} \tag{4}$$

$$\sigma_{\theta\theta} + i\tau_{r\theta} = \phi'_j + \overline{\phi'_j} + [\bar{z}\phi''_j + \Psi'_j]e^{2i\theta}. \tag{5}$$

It is assumed that as a result of the expansive reactions, radius r_i expands uniformly with $u_r(r_i) = \bar{u}_r^{\,i}$ and radius r_o expands uniformly with $u_r(r_o) = \bar{u}_o^{\,i}$. With these artificial boundary conditions at the interfaces of the different materials, the problem can be separated into three boundary value problems.

Next, it is convenient to choose the two complex functions in the form $\phi(z) = \sum_{j=-N}^{M} A_j z^j$ and $\Psi(z) = \sum_{j=-N}^{M} B_j z^j$. From the positive and negative powers of z in the expansions for $\phi(z)$ and $\Psi(z)$, we need to choose only a few for our problem. This is done below separately for the three domains.

2.1. Solution for Domain 1

The complex functions in Domain 1 are given by the following Eq. (6):

$$\phi_1(z) = a_1 z = a_1 re^{i\theta}, \quad \Psi_1(z) = b_1 z = b_1 re^{i\theta}. \tag{6}$$

Using $\overline{\Psi_1(z)} = \overline{b_1} re^{-i\theta}$ and $\phi'_1 = a_1$, the displacement field becomes Eqs. (7,8):

$$u_r + iu_\theta = \frac{1}{2\mu_1} e^{-i\theta}\left[\kappa_1 a_1 re^{i\theta} - \overline{a}_1 re^{i\theta} - \overline{b}_1 re^{-i\theta}\right] \tag{7}$$

$$= \frac{r}{2\mu_1}\left[\kappa_1 a_1 - \overline{a}_1 - \overline{b}_1 e^{-2i\theta}\right]. \tag{8}$$

Assuming that the solution does not depend on θ, then this leads to $b_1 = 0$ and $a_1 = a$, where a is real. Therefore, the displacement field in Domain 1 is given by Eq. (9):

$$u_r + iu_\theta = \frac{1}{2\mu_1}[\kappa_1 - 1]a\,r. \tag{9}$$

From the boundary condition [Eq. (10)],

$$u_r(r = r_i) = \frac{1}{2\mu_1}[\chi_1 - 1]a \overset{!}{=} \overline{u}_r^{\,i} \tag{10}$$

it follows that of [Eq. (11)]

$$a = \frac{2\mu_1 \overline{u}_r^{\,i}}{[\kappa_1 - 1]r_i.} \tag{11}$$

Therefore, the stresses in Domain 1 due to the boundary condition at $r = r_i$ are Eq. (12)

$$\sigma_{rr} = \sigma_{\theta\theta} = 2a = \frac{4\mu_1\, \overline{u}_r^{\,i}}{[\kappa_1 - 1]r_i.} \tag{12}$$

2.2. Solution for Domain 2

If we use the complex functions [Eqs. (13,14)]

$$\phi_2(z) = c_1 z, \quad \Psi_2(z) = d_{-1}z^{-1} \tag{13}$$

$$\Psi'_2(z) = -d_{-1}z^{-2} = -d_{-1}r^{-2}e^{-2i\theta} \tag{14}$$

then, the displacement field is given by Eqs. (15–17):

$$2\mu_2(u_r + iu_\theta) = e^{-i\theta}\left[\kappa_2 c_1 r e^{i\theta} - r e^{i\theta}\bar{c}_1 - \bar{d}_{-1} r^{-1} e^{i\theta}\right] \quad (15)$$

$$\text{or} = r[k_2 c_1 - \bar{c}_1] - \frac{1}{r}\bar{d}_{-1} \quad (16)$$

$$2\mu_2(u_r + iu_\theta) = r\{\kappa_2 \mathrm{Re}[c_1] + i\kappa_1 \mathrm{Im}[c_1] - \mathrm{Re}[c_1] + i\mathrm{Im}[c_1]\} + \frac{1}{r}\{\mathrm{Re}[d_{-1}] - i\mathrm{Im}[d_{-1}]\}. \quad (17)$$

Because of rotational symmetry $\mathrm{Im}[c_1] = 0$, $\mathrm{Im}[d_{-1}] = 0$, $c_1 = c$, $d_{-1} = d$, where c and d are real coefficients. Therefore, in Domain 2, the displacements can be written as Eqs. (18,19):

$$2\mu_2(u_r + iu_\theta) = r[\kappa_2 - 1]c - \frac{1}{r}d \quad (18)$$

or,

$$u_r = \frac{1}{2\mu_2}\left[r(\kappa_2 - 1)c - \frac{1}{r}d\right], \quad u_\theta = 0 \quad (19)$$

whereby, we obtain the stresses in the following Eqs. (20–22):

$$\sigma_{rr} - i\tau_{r\theta} = \phi_2' + \overline{\phi_2'} - [\bar{z}\phi_2'' + \Psi_2']e^{2i\theta} \quad (20)$$

$$or = 2c + \frac{1}{r^2}d \quad (21)$$

$$\sigma_{\theta\theta} + i\tau_{r\theta} = 2c - \frac{1}{r^2}d. \quad (22)$$

The boundary conditions [see Eqs. (23–24)]

$$u_r(r = r_i) = \frac{1}{2\mu_2}\left[r_i(\kappa_2 - 1)c - \frac{1}{r_i}d\right] \stackrel{!}{=} \bar{u}_r^i \quad (23)$$

$$u_r(r = r_0) = \frac{1}{2\mu_2}\left[r_0(\kappa_2 - 1)c - \frac{1}{r_0}d\right] \stackrel{!}{=} \bar{u}_r^o \quad (24)$$

allow the determination of the coefficients c and d [see Eqs. (25,26)]

$$c = \frac{2\mu_2[r_0\bar{u}_r^0 - r_i\bar{u}_r^i]}{(\kappa_2 - 1)(r_0 - r_i)(r_0 + r_i)} \quad (25)$$

$$d = \frac{2\mu_2 r_0 - r_i[r_i\bar{u}_r^0 - r_0\bar{u}_r^i]}{(r_0 - r_i)(r_0 + r_i).} \quad (26)$$

Finally, the stresses at $r = r_i$ are given by Eqs. (27,28)

$$\sigma_{rr}(r_i) = \frac{2\mu_2\left[\kappa_2 r_i r_0\bar{u}_r^0 + r_i r_0\bar{u}_r^0 - \kappa_2 r_0^2\bar{u}_r^0 + r_0^2\bar{u}_r^i - 2r_i^2\bar{u}_r^i\right]}{(\kappa_2 - 1)r_i(r_0 - r_i)(r_0 + r_i)} \quad (27)$$

$$\sigma_{\theta\theta}(r_i) = \frac{-2\mu_2\left[\kappa_2 r_i r_0\bar{u}_r^0 - 3r_i r_0\bar{u}_r^0 - \kappa_2 r_0^2\bar{u}_r^i + r_0^2\bar{u}_r^i + 2r_j^2\bar{u}_r^i\right]}{(\kappa_2 - 1)r_i(r_0 - r_i)(r_0 + r_i).} \quad (28)$$

2.3. Solution for Domain 3

The model assumes that the cross-sectional area of the inclusion is sufficiently distant from an outer boundary; in other words, areas 1 and 2 lie within the infinite Domain 3. It also assumes that the stresses vanish for r going to infinity. The complex functions for Domain 3 are in the form $\phi_3 = 0$, $\Psi_3(z) = fz^{-1}$ where f is a real coefficient. This leads to the following displacement field [Eq. (29)]:

$$u_r + iu_\theta = \frac{1}{2\mu_3}\left[-\frac{1}{r}f\right]. \quad (29)$$

The boundary condition [see Eq. (30)]

$$u_r(r = r_0) = \frac{-1}{2\mu_3 r_0}f = \bar{u}_r^0 \quad (30)$$

allows the determination of the coefficient f [see Eq. (31)]

$$f = -2\ \mu_3 r_0\bar{u}_r^0 \quad (31)$$

Therefore, stresses in Domain 3 are Eqs. (32–34):

$$\sigma_{rr} = \frac{1}{r^2}f = -2\mu_3 r_0\bar{u}_r^0\frac{1}{r^2} \quad (32)$$

$$\sigma_{\theta\theta} = -\frac{1}{r^2}f = 2\mu_3 r_0\bar{u}_r^0\frac{1}{r^2} \quad (33)$$

$$\tau_{r\theta} = 0. \quad (34)$$

In summary, the solution was obtained based on the following assumptions: (a) the effect of the expansion in Domain 1 may be modeled by prescribing a uniform radial displacement; (b) stresses at infinity are zero, i.e., there is no influence from boundaries at a finite distance. (This last assumption may not be realistic for some problems.) The solution for Domain 3 with finite dimensions would prove to be much more complicated. Such analysis would need the geometry of the boundary and loading conditions at the boundary.

3. Applications

3.1. Corrosion of reinforced concrete

The corrosion mechanism of reinforcing bars decreases the original cross-section of the bar, but generates corrosion products that occupy greater volume than the reinforcing bars did initially. Therefore, it is useful to define

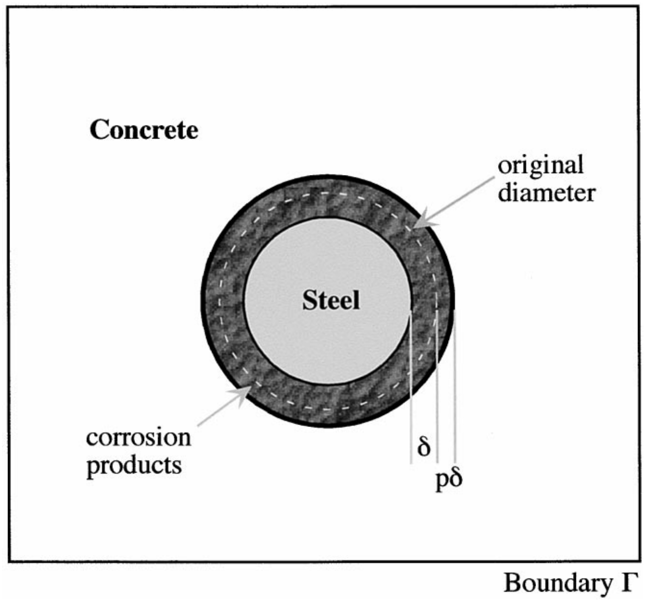

Fig. 2. Model used to estimate the stresses developed during the corrosion of reinforced concrete. The corrosion products are expansive and generate tensile hoop stresses in the matrix.

a "coefficient of expansion," p, which relates the penetration of corrosion, δ, to the expansion of the corrosion products (Fig. 2). The tensile hoop stress at the corrosion products–concrete interface $r = r_o$ is given by Eq. (33) [Eq. (35)]:

$$\sigma_{\theta\theta} = 2\mu_3 \frac{\bar{u}_r^0}{r_0}. \tag{35}$$

The displacement at the interface between the corrosion products and the concrete is Eq. (36):

$$\bar{u}_r^0 = p\delta. \tag{36}$$

Therefore,

$$\sigma_{\theta\theta} = 2\mu_3 \frac{p\delta}{r_0}. \tag{37}$$

The reacted mass of steel, m, during the corrosion process can be determined by Faraday's equation [see Eq. (38)]:

$$m = \frac{Ita}{nF} \tag{38}$$

where I is the current, t is the time, a is the atomic mass, n is the equivalents exchanged, and F is the Faraday's constant. The corrosion rate r can be obtained by dividing Eq. (37) by t and by the surface area A [Eq. (39)]:

$$r = \frac{ia}{nF} \tag{39}$$

where i, the current density, is given by I/A. The rate of penetration, $\dot{\delta}$, during the corrosion process can be determined from dividing Eq. (39) by the density of the metal. For steel (n = 2), the equation becomes:

$$\dot{\delta} = 0.0116i \qquad \text{(mm/year)} \tag{40}$$

with i in μA/cm^2. Integrating Eq. (40), Eq. (41) becomes:

$$\delta = 0.0116it \tag{41}$$

with time, t, in years. Eq. (37) becomes Eq. (42):

$$\sigma_{\theta\theta} = \frac{0.0116Epit}{(1+\nu)r_0} \tag{42}$$

where E and ν are Young's modulus and Poisson's ratio of concrete, respectively. Assuming that the first crack develops at time t_c when the hoop stress reaches the concrete tensile strength, f_t:

$$t_c = \frac{f_t(1+\nu)r_0}{0.0116Epi.} \tag{43}$$

The time for first crack t_c is proportional to the bar radius and inversely proportional to the current density. Both dependences were verified in field conditions. The difficulty of using Eq. (43) for the prediction of the life cycle of a structure is the uncertainty of the coefficient of expansion, p. Even if the composition of the corrosion products were known thereby establishing its potential volumetric expansion, the value of the expansion would still be unknown because concrete is a porous material and a portion of the corrosion products diffuse away. The value of p depends, therefore, not only on the stoichiometry of the corrosion products, but also on the permeability of the matrix. Experimental tests should be conducted to calibrate the value of p for a given concrete mixture.

3.2. Ice formation in concrete

To design durable concrete structures, it is important to understand the stress and strain levels generated during the freezing of concrete. Recently, there has been much research conducted on the behavior of high-strength concrete exposed to freeze and thaw cycles. Even though high-strength concrete has small amounts of freezable water, the matrix has very low permeability; if even only a small amount of ice is formed, there will be large tensile stresses enough to crack the matrix.

Fagerlund [5] calculated the intensity of these stresses using "closed container models," where cement paste surrounds a hole filled with freezable water and unfreezable water. His analysis confirms that saturated concrete containing even a small amount of freezable water cannot withstand freezing unless air voids are present to relieve the stress.

Assuming that the freezable water is located in a hole and the matrix surrounds it, the present model predicts that after

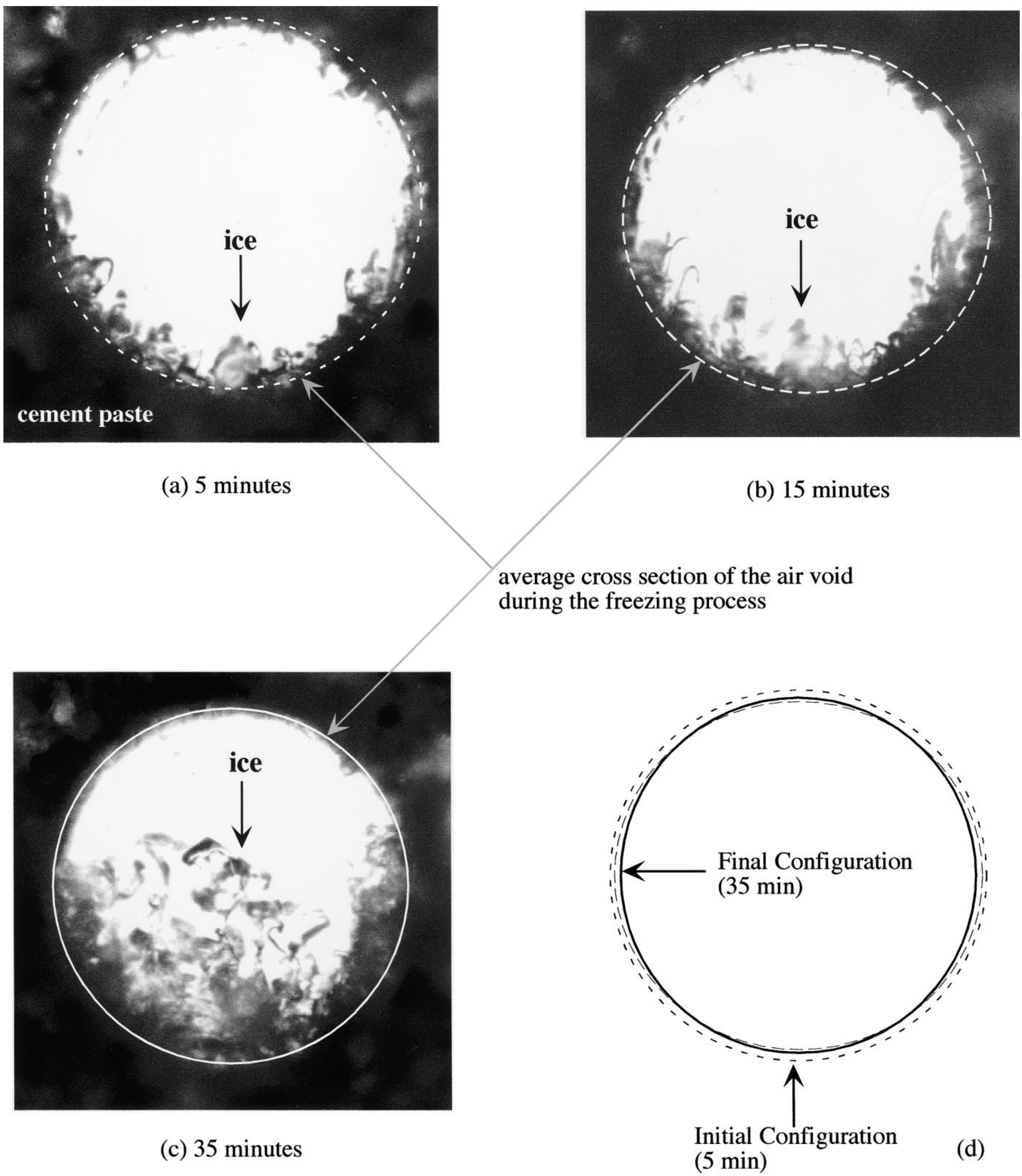

Fig. 3. Determination of the change of dimension in an air void during freezing of concrete (a–c). Note the decrease in the air void diameter (d), indicating shrinkage in the cement paste matrix.

freezing, the stresses in the ice become [see Eq. (44)]

$$\sigma_{rr} = \sigma_{\theta\theta} = 2a = \frac{4\mu_{\rm ice}\overline{u}_{\rm r}^{i}}{[\kappa_{\rm ice} - 1]r_i} \tag{44}$$

and the stresses in the matrix are as follows in Eqs. (45–47):

$$\sigma_{rr} = -2\ \mu_{\rm matrix}\ r_0\ \overline{u}_{\rm r}^{0}\,\frac{1}{r^2} \tag{45}$$

$$\sigma_{\theta\theta} = 2\mu_{\mathrm{matrix}} r_0\, \bar{u}_r^0 \frac{1}{r^2} \tag{46}$$

$$\tau_{r\theta} = 0. \tag{47}$$

There are basically two problems with this approach. First, Eq. (46) indicates that the cement paste matrix would not be able to withstand any freezing of the water. Assuming a tensile strength of 4 MPa and a shear modulus of 16 GPa, the maximum strain that the cement paste could take would be in the order of 0.01%, far less than required when water freezes. Second, and more important, the thermodynamics of the freezing process in porous media indicate that freezing water in the cement paste moves towards, not from, the capillary pores. This leads to a shrinkage in the matrix.

Fig. 3 verifies this assumption, showing the sequence of ice propagation in the smaller pores. The images were obtained using the "directional solidification" (Wang et al. [6]), which permits the controlled cooling and warming of a relatively large sample. The amount of time after the freezing front passed is indicated in each of the images, and the external diameter of the air void is shown in Fig. 3. There is a clear decrease of air void diameter as freezing continues in the matrix, indicating a shrinkage of the matrix, as expected by the thermodynamics models.

It is possible to use these dimensional changes to determine the strain and stress field in the matrix. It is common practice in rock mechanics to excavate a borehole to determine the strain field in the rock mass (Jaeger and Cook [7]). In air-entrained concrete, the air void can be used as a natural strain gage. Once the displacement field is determined, as shown in Fig. 3d, it is possible to establish the shrinkage in the matrix by assuming an uncoupled, quasi-static thermo-elastic theory, with a radial temperature variation, $T = T(r)$. The radial displacement $u_r = u_r(r)$ has the following form [see Eq. (48)] for plane strain conditions in Domains 2 and 3 [8]:

$$u_r = \alpha \frac{(1+\nu)}{(1-\nu)} \frac{1}{r} \int_a^r rT(r)\mathrm{d}r + Ar + \frac{B}{r} \qquad [2-\mathrm{d\ model}] \tag{48}$$

where $a = r_i$ for Domain 2 and $a = r_o$ for Domain 3.

For a given temperature distribution, $T(r)$, the coefficients A and B can be computed from the boundary conditions in Domains 2 and 3 as done earlier, and the stresses can be obtained. In addition to the $1/r$-term and the r-term, the displacement u_r includes an integral involving the temperature distribution $T(r)$.

Note that we considered a plane model in cylindrical coordinates in our discussion. For a hollow sphere model with rotational symmetry, the radial displacement u_r can be expressed, depending on a temperature distribution $T(r)$, in the following form [Eq. (49)]:

$$u_r = \alpha \frac{(1+\nu)}{(1-\nu)} \frac{1}{r^2} \int_a^r r^2 T(r)\mathrm{d}r + Ar + \frac{B}{r^2} \qquad [3-\mathrm{d\ model.}] \tag{49}$$

In spherical coordinates, we have the term $1/r^2$, whereas in cylindrical coordinates, the displacement u_r involves the term $1/r$. Again, the free coefficients A and B can be computed from the displacement boundary conditions on $r = r_i$ and $r = r_o$ for Domain 2 and from conditions on $r = r_o$ and $r \to \infty$ for Domain 3. With the only nonvanishing displacement component u_r the stresses can be computed easily.

4. Conclusions

Muskhelishvili's method was used to determine the stress and strain fields caused by the expansion of one of the phases in concrete. These equations were used in conjunction with Faraday's equation to predict when cracking will occur as a result of corrosion of reinforcing bars. Experimental work is necessary to calibrate the life prediction based on this model.

The stress equations were also used to predict the tensile stresses caused by ice formation in a pore. It is proposed to use the air void as a strain gage during ice formation in cement paste or concrete. The changes in dimension of the air void can be used to quantify the intensity of shrinkage in the matrix and to determine the existence of thermal and mechanical gradients close to the air void.

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