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Failure probability prediction of concrete components

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Abstract

In order to predict the probability of failure for brittle fracture of concrete components under multiaxial stress states, the imperfections of concrete components are modeled as cracks with different shapes in this paper. A new probability distribution function for evaluating the failure probability of concrete components is proposed. A simplified measurement method for determining the parameters of the governing Weibull distribution, using the three-point bending test, is presented and discussed. The experimental results of the combined bending/torsion failure tests of concrete components verify that the proposed crack model is more reasonable than the Batdorf's crack model and the proposed prediction formula can evaluate the failure probability of concrete components accurately.

Keywords: Cement; Concrete; Microcracking; Finite element analysis

1. Introduction

Structural engineering is generally based on the strength-of-materials theories. It is assumed that failure of a component occurs as soon as its limit strength is exceeded. However, for materials such as rock, concrete and ceramics, in general, large variations in strength were observed, especially if the number of specimens is small. Therefore, the concept of conventional limit strength may not be sufficient as strength criterion for many cases. The optimal way to define the limit strength of these kinds of brittle materials should be by means of statistics. In other words, the failure criterion of the material should be expressed in a probabilistic form. A designer, therefore, must be able to assure himself/herself of an acceptable low probability of failure during service when these materials are used in structures.

As is well known, the real strength of concrete components shows a much smaller strength than expected and often differs from sample to sample. Hsu et al. [1] was among the first to recognize the presence of the initial microcracks at the interface between aggregate and matrix. The growth of microcracks results in the real strength of concrete smaller than expected [2]. The randomness of the locations, shapes and orientations of cracks leads to a large variation of the limit strength of materials [3–5]. The first significant contribution to the investigation of the failure probability of brittle materials was made by Weibull [6], and his theories are still the basis for most analysis in this field. In Weibull's theory, stressed brittle structure is modeled as a loaded chain and failure occurs when its weakest link is broken. Similarly, from the statistical fracture mechanics point of view, the imperfections of material are modeled as cracks, when an isolated crack in an elastic body is loaded normal to its plane, it will become unstable and grow catastrophically, resulting in fracture. The fracture stress of the entire body is that of the weakest crack. Batdorf and Crose [7] and Batdorf [8] investigated the statistical fracture characteristics of brittle materials, and pointed out that the effective stress σ_e causing fracture is a function of normal stress σ_n and shear stress τ_n which are applied to the crack plane and result in very high local stress. For a material containing cracks of only a single plane form, the cumulative probability of failure was expressed as Weibull's two-parameter or three-parameter distribution. In terms of the Batdorf's theory, Li and Fang [9], based on their failure tests of granite specimens, investigated the probabilistic distribution of the material strength and proposed a probabilistic strength criterion for evaluating the failure probability of rock material. However, in the studies mentioned above, the imperfections of materials were modeled as cracks with identical shape. Obviously, this is not true for describing the

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variation of the imperfections. In order to take the effects of the variation of the microcrack shapes on concrete strength into account, it is necessary to modify Batdorf's crack model. In this paper, the imperfections existing in concrete materials are modeled as several different kinds of cracks. Based on Batdorf's statistical fracture theory, the prediction formula of failure probability of concrete components under multiaxial stress state is established.

This study focuses on the failure probability prediction of plain concrete components. In fact, plain concrete materials are widely used in engineering practices. In addition, the present research work on plain concrete provides a foundation for the future works on coarse aggregate added concretes and furthermore reinforced concretes.

2. Distribution function of failure probability

2.1. Batdorf's failure probabilistic distribution

The failure probability, P_f, of a brittle material component, under the multiaxial stress state, can be expressed as [8]

$$P_f = 1 - \exp\left[-\int_V n(\Sigma)\mathrm{d}V\right] \quad (1)$$

where $n(\Sigma)$ is failure probability of unit volume (unit area), V is the volume (surface area) of the component. Based on the weakest link assumption, the failure probability is fitted by Weibull's distribution, and $n(\Sigma)$ is expressed as

$$n(\Sigma) = \begin{cases} \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{\sigma_e}{\sigma_0}\right)^m \sin\varphi \mathrm{d}\varphi \mathrm{d}\theta & \text{(body cracks)} \\ \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\sigma_e}{\sigma_0}\right)^m \mathrm{d}\varphi & \text{(surface cracks)} \end{cases} \quad (2)$$

where φ and θ are the angles between the line normal to the crack surface and the principal axes of stress (Fig. 1),

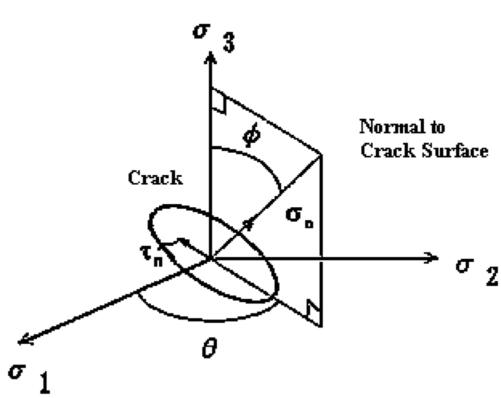

Fig. 1. Angles of crack surface.

respectively; m and σ_0 are Weibull parameters under the equitriaxial stress state; σ_e is the equivalent stress, and can be expressed as

$$\sigma_e = (\sigma_n^2 + D\tau_n^2)^{1/2} \quad (3)$$

where σ_n and τ_n (Fig. 1) are the normal stress and shear stress on the surface of the crack, i.e.

$$\sigma_n = \sigma_1\cos^2\varphi + \sigma_2\sin^2\varphi\cos^2\theta + \sigma_3\sin^2\varphi\sin^2\theta$$

$$\tau_n = [\sigma_1^2\cos^2\varphi + \sigma_2^2\sin^2\varphi\cos^2\theta + \sigma_3^2\sin^2\varphi\sin^2\theta - \sigma_n^2]^{1/2} \quad (4)$$

D is a constant that is related to the fracture criterion and the shape of the crack. In general, D can be calculated by the following equation

$$D = \sqrt{\frac{1}{(1-0.5v)^2}} \quad (5)$$

where v is the Poisson's ratio.

If the microcrack is modeled as a penny-shaped body crack or a half penny-shaped surface crack and the energy release rate criterion is used, for equitriaxial stress state, $\sigma_n = \sigma_1 = \sigma_2 = \sigma_3 = \sigma$, $\tau_n = 0$. $\sigma_e = \sigma$, Eq. (1) can be rewritten as as

$$P_f = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m V\right] \quad (6a)$$

or

$$1 - P_f = \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m V\right] \quad (6b)$$

Taking natural logarithm for the both sides of Eq. (6b) twice, one obtains

$$\ln\ln(1-P_f)^{-1} = m\ln\sigma + \ln\left(\frac{V}{\sigma_0^m}\right) \quad (7)$$

It can be seen, from Eq. (7), that the relationship between the double logarithm of the reliability to minus one power and the logarithm of the stress is linear. Therefore, by means of the equitriaxial test, the Weibull parameters of the concrete component are

$$\left.\begin{array}{l} m = \text{slope} \\ \sigma_0 = [V\mathrm{e}^{-(\text{intercept})}]^{1/m} \end{array}\right\} \quad (8)$$

In general, instead of the equitriaxial stress state discussed above, the Weibull parameters can be determined by the alternative stress state test indirectly.

2.2. Modified Batdorf's failure probabilistic distribution

A number of failure tests of concrete components [10] show that the relationship of ln ln $(1-P_f)^{-1}$ and ln σ is nonlinear. Eq. (7), therefore, is not suitable to fit this nonlinear relationship. The mechanism of the nonlinear relationship may be explained as follows. In general, the shapes of different imperfections in the body or on the surface of the component are different. These imperfections can be modeled as cracks with different shapes. From a fracture mechanics point of view, the critical stress of unstable crack propagation differs from crack to crack. Therefore, at different stress levels, the strength of the component may be dominated by the propagation of different types of crack. The variety of cracks leads to the nonlinear relationship between ln ln $(1-P_f)^{-1}$ and ln σ [10]. In order to describe this nonlinear relation, it is quite straightforward to model the imperfections of the component as several types of crack with different shapes, so that the curve can be fitted by broken lines. To establish the failure probabilistic distribution function, the following assumptions are made

(1) The imperfections existing in the concrete components can be modeled as cracks with different shapes;
(2) The spatial distribution of every crack is independent and interaction between the cracks is negligible for the sake of simplicity;
(3) The failure probability caused by unstable propagation of any type of crack obeys two-parameter Weibull's distribution.

Suppose that P_{fi} represents the failure probability, which is caused by the unstable propagation of the ith type crack, then the probability of no failure P_{si} is

$$P_{si} = 1 - P_{fi} \tag{9}$$

From the above assumptions (1) and (2), the probability of no failure of a concrete component P_s can be expressed as

$$P_s = \prod_{i=1}^{N}(1 - P_{fi}) \tag{10}$$

where N is the number of crack types.

If the imperfections are modeled as r types of body crack and $(N-r)$ types of surface crack, Eq. (10) can be rewritten as

$$P_s = \prod_{i=1}^{r}(1 - P_{fi}) \prod_{j=1}^{N-r}(1 - P_{fj}) \tag{11}$$

Substituting Eq. (2) and Eq. (3) into Eq. (11), one obtains

$$P_s = \exp\left[-\int_V \sum_{i=1}^{r} K_i(n_{12}, n_{13}, m_i)\left(\frac{\sigma_1}{\sigma_{0i}}\right)^{m_i} dV\right] \times \exp\left[-\int_A \sum_{j=1}^{N-r} K_j(n_{12}, m_j)\left(\frac{\sigma_1}{\sigma_{0j}}\right)^{m_j} dA\right] \tag{12a}$$

where V and A are the volume and surface area of the component, respectively.
or

$$P_f = 1 - P_s \tag{12b}$$

where σ_{0i}, m_i, σ_{0j} and m_j are the Weibull parameters corresponding to the ith type crack, and

$$n_{12} = \sigma_2/\sigma_1, \quad n_{13} = \sigma_3/\sigma_1 \text{ for } \sigma_1 > \sigma_2 > \sigma_3 \tag{13a}$$

$$K_i(n_{12}, n_{13}, m_i) = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2}[D_i f_2 - (D_i - 1) f_1]^{m_i/2} \times \sin\varphi \, d\varphi \, d\theta \tag{13b}$$

$$K_i(n_{13}, m_j) = \frac{2}{\pi}\int_0^{\pi/2}[D_j f_2' - (D_j - 1) f_1']^{m_j/2} d\varphi \tag{13c}$$

where D_i and D_j are constants which are similar to those in Eq. (5).

$$f_1 = (\cos^2\varphi + n_{12}\sin^2\varphi\cos^2\theta + n_{13}\sin^2\varphi\sin^2\theta)^2 \tag{14a}$$

$$f_2 = \cos^2\varphi + n_{12}\sin^2\varphi\cos^2\theta + n_{13}\sin^2\varphi\sin^2\theta \tag{14b}$$

$$f_1' = (\cos^2\varphi + n_{12}\sin^2\varphi\cos^2\theta)^2 \tag{14c}$$

$$f_2' = \cos^2\varphi + n_{12}\sin^2\varphi\cos^2\theta \tag{14d}$$

3. Weibull parameters of concrete components

The failure probability $P_f(\sigma_{Mi})$ of concrete specimens can be determined experimentally by [11]

$$P_f = (\sigma_{Mi}) = \frac{i}{n+1} \quad (i = 1, 2, \cdots n) \tag{15}$$

where n is the number of the specimens, σ_{Mi} is the observed fracture stress, for the three-point bending test, σ_{Mi} is the maximum tension stress of the specimen, at which the specimen is failure. i is the number of failure specimens at the observed fracture stress σ_{Mi}.

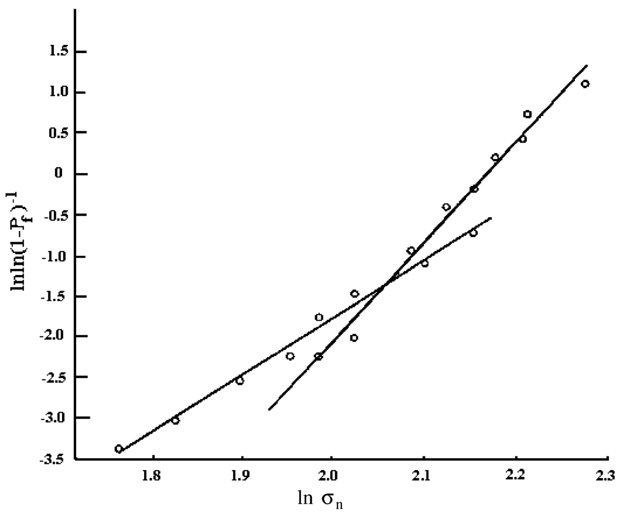

Fig. 2. Regression for Weibull parameters.

Arranging σ_{Mi} in ascending order, i.e.

$$\sigma_{Mi-1} < \sigma_{Mi} < \sigma_{Mi+1} \tag{16}$$

The parameters m, σ_0 can be calculated by Eq. (8) if the imperfection is modeled as one type of crack.

It is noted, from Eq. (12a), that the linear relationship between the double logarithm of the probability of no failure to the power of minus one and the logarithm of the observed stress is not held. Therefore, the linear regression cannot be directly used to determine the Weibull parameters. According to the theory of fracture mechanics, however, under the same stress state, the critical stress is different for different types of cracks. It is reasonable, therefore, to infer that the failure of the component is mainly caused by the unstable propagation of one type of crack in a certain stress range, i.e., the fracture strength of the component is dominated by one type of crack in such a stress range, the effect of other types of crack on fracture is negligible. The linear relationship which is similar to Eq. (8) is approximately satisfied in each stress range. Thus, the broken lines can be used to fit the experimental data, and then the Weibull parameters corresponding to different types of cracks can be determined.

The three-point bending test is a convenient way for the determination of the Weibull parameters. Letting the length, width and height of the specimen be l, b and h, respectively, from Eq. (2), the failure probability of the specimen, dominated by the ith type of crack, can be expressed as

$$P_{fi} = \begin{cases} 1 - \exp\left[-\dfrac{u_1(m_i)lbh}{2(m_i+1)^2}\left(\dfrac{\sigma_{mi}}{\sigma_{0i}}\right)^{m_i}\right] & \text{(body crack)} \\ 1 - \exp\left\{ -\dfrac{2u_1(m_i)}{\pi}\left[\dfrac{lb}{(m_i+1)^2} + \dfrac{lh}{m_i+1}\right]\left(\dfrac{\sigma_{Mi}}{\sigma_{0i}}\right)^{m_i}\right\} & \text{(surface crack)} \end{cases} \tag{17}$$

where

$$u_1(m_i) = \int_0^{\pi/2} (\cos^4\varphi + D_i\sin^2\varphi\cos^2\varphi)^{m_i/2}\sin\varphi \mathrm{d}\varphi \tag{18a}$$

$$u_2(m_i) = \int_0^{\pi/2} (\cos^4\varphi + D_i\sin^2\varphi\cos^2\varphi)^{m_i/2}\mathrm{d}\varphi \tag{18b}$$

Similar to Eq. (8), Eq. (17) can be rewritten as

$$\ln\ln(1-P_{fi})^{-1} = m_i\ln\sigma_{Mi} + \ln\left[\frac{lbhu_1(m_i)}{2(m_i+1)^2\sigma_{0i}^{m_i}}\right] \quad \text{(body crack)} \tag{19a}$$

and

$$\ln\ln(1-P_{fi})^{-1} = m_i\ln\sigma_{Mi} + \ln\left\{\frac{2u_2(m_i)}{\pi\sigma_{0i}^{m_i}}\left[\frac{lb}{(m_i+1)^2} + \frac{lh}{m_i+1}\right]\right\} \quad \text{(surface crack)} \tag{19b}$$

or it is expressed as a unified form

$$\ln\ln(1-P_f)^{-1} = A\ln\sigma_M + B \tag{20}$$

Eq. (20) is a linear equation. By comparison of Eq. (20) with Eqs. (19a) and (19b), the Weibull parameters corresponding to any type of crack are

$$\left.\begin{aligned} m_i &= A \\ \sigma_{0i} &= \left[\frac{lbhu_1(m_i)}{2(m_i+1)^2}\right]^{1/m_i}\exp(-Bm_i) \end{aligned}\right\} \text{(body crack)} \tag{21}$$

$$\left.\begin{aligned} m_i &= A \\ \sigma_{0i} &= \left\{\frac{2u_2(m_i)}{\pi}\left[\frac{lb}{(m_i+1)^2} + \frac{lh}{m_i+1}\right]\right\}^{1/m_i}\exp(-Bm_i) \end{aligned}\right\} \text{(surface crack)} \tag{22}$$

4. Experimental results

The concrete specimens are made of cement–sand ratio of 4 to 6, the characteristic dimension of sand is 0.15–0.5 mm. The dimension of the specimen is $30 \times 30 \times 300$ mm. The failure test was performed for 100 such specimens.

Table 1
The Weibull parameters

Crack type	m_i	σ_{0i} (MPa)	Correlation coefficient
Body crack	7.081	1.118	.998
Surface crack	12.118	3.753	.996

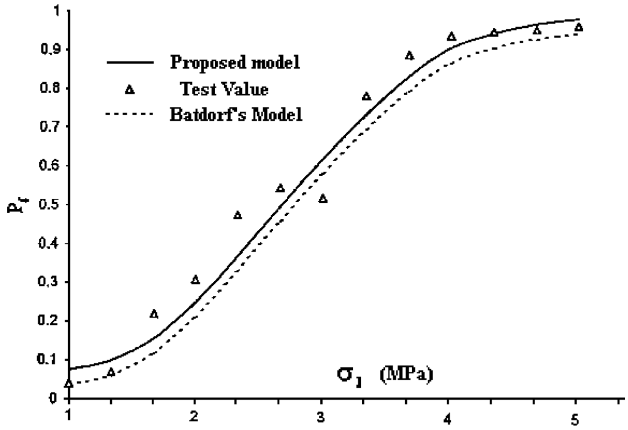

Fig. 3. Evaluation of failure probability.

From the observations of fracture surfaces, it was found that the imperfections of the specimens can be approximately modeled by two types of crack, one is the penny-shaped body crack and another is the half penny-shaped surface crack. The regression curve is shown in Fig. 2 and the Weibull parameters are given in Table 1.

5. Prediction of failure probability

The bending/torsion combined experiment, in which the bending and torsion moments are applied at the same time on the free end of the specimen—a cantilever beam, has been carried out for the concrete specimens with identical dimensions and properties of each specimen to verify the proposed failure probability distribution. The failure probability of the specimens is calculated by Eq. (15) and the failure stress is determined by the finite element method (FEM). We let $N=2$, $r=1$ in Eq. (12a). If the volume of an element is small enough, the integral can be replaced by summation. The failure probability, can be expressed as

$$P_{\mathrm{f}} = 1 - \exp\left\{ -\sum_{i=1}^{N_{\mathrm{e}}} K_i[(n_{12})_i, (n_{13})_i, m_1] \left(\frac{\sigma_{1i}}{\sigma_{01}} \right)^{m_1} V_i \right\} \times \exp\left\{ -\sum_{j=1}^{N_{\mathrm{s}}} K_j[(n_{12})_j, m_2] \left(\frac{\sigma_{1j}}{\sigma_{02}} \right)^{m_2} A_j \right\} \tag{23}$$

where N_{e} is the number of the total elements and N_{s} is the number of the elements which constitute the surface of the specimen, V_i and A_j are the volume and surface area of an element, respectively.

Fig. 3 shows the prediction results of the failure probability obtained by Eq. (23). For comparison purposes, the results calculated based on the assumption of one type crack (Batdorf's crack model) are plotted in the same figure. It can be seen that the values of the failure probability calculated by Eq. (23) are closer to the experimental data. This illustrates that the proposed crack model is more reasonable than Batdorf's crack model. On the other hand, the probabilities calculated by Eq. (23) are greater than those obtained from Batdorf's model, suggesting that the results obtained by the proposed model are conservative.

6. Conclusions

This paper deals with failure probability prediction of concrete components. Based on the failure tests of concrete components and the statistical theory of fracture mechanics, the imperfections of concrete materials are modeled as body and surface cracks with penny and half-penny shapes and a new prediction formula for failure probability of concrete components under multiaxial stress state has been proposed. The regressive formulae for determining Weibull parameters by the three-point bending test have been deduced, which significantly simplify the measurement procedure for determining the Weibull parameters. The experimental results of the combined bending/torsion failure test of the plain concrete components have shown that the proposed crack model is more reasonable than Batdorf's crack model and the proposed prediction formula can evaluate the failure probability of concrete components accurately.

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