Available online at www.sciencedirect.com
ScienceDirect
Ceramics International 35 (2009) 237–246
CERAMICS INTERNATIONAL
www.elsevier.com/locate/ceramint

Is Weibull distribution the most appropriate statistical strength distribution for brittle materials?

Bikramjit Basu [a,*], Devesh Tiwari [b], Debasis Kundu [c], Rajesh Prasad [a,1]

[a] Department of Materials and Metallurgical Engineering, Indian Institute of Technology Kanpur, Pin 208016, India
[b] Department of Computer Science and Engineering, Indian Institute of Technology Kanpur, Pin 208016, India
[c] Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India

Received 13 April 2007; received in revised form 29 August 2007; accepted 4 October 2007
Available online 22 January 2008

Abstract

Strength reliability, one of the critical factors restricting wider use of brittle materials in various structural applications, is commonly characterized by Weibull strength distribution function. In the present work, the detailed statistical analysis of the strength data is carried out using a larger class of probability models including Weibull, normal, log-normal, gamma and generalized exponential distributions. Our analysis is validated using the strength data, measured with a number of structural ceramic materials and a glass material. An important implication of the present study is that the gamma or log-normal distribution function, in contrast to Weibull distribution, may describe more appropriately, in certain cases, the experimentally measured strength data.
© 2007 Elsevier Ltd and Techna Group S.r.l. All rights reserved.

Keywords: Two-parameter distributions; Kolmogorov statistics; χ^2statistics; Shape parameter; Scale parameter; Strength data

1. Introduction

Brittle materials, like ceramics have many useful properties like high hardness, stiffness and elastic modulus, wear resistance, high strength retention at elevated temperatures, corrosion resistance associated with chemical inertness, etc. [1]. The advancement of ceramic science in the last few decades has enabled the application of this class of materials to evolve from more traditional applications (sanitary wares, pottery, etc.) to cutting edge technologies, including rocket engine nozzles, engine parts, implant materials for biomedical applications, heat resistant tiles for space shuttle, nuclear materials, storage and renewable energy devices, fiber optics for high speed communications and elements for integrated electronics like micro-electro-mechanical systems (MEMS).

In many of the engineering applications requiring load bearing capability i.e. structural applications, it has been realized over the years that an optimum combination of high toughness with high hardness and strength reliability is required [2]. Despite having much better hardness compared to conventional metallic materials, the major limitations of ceramics for structural and specific non-structural applications are the poor toughness and low strength reliability [3]. The poor reliability in strength or rather large variability in strength property of ceramics is primarily due to the variability in distribution of crack size, shape and orientation with respect to the tensile loading axis [4]. Consequently, the strength of identical ceramic specimens under identical loading conditions is different for a given ceramic composition. The physics of the fracture of brittle solids and the origin of Weibull's strength theory is discussed in some details in Section 2.

The above mentioned limitations have triggered extensive research activities in the ceramic community to explore several toughening mechanisms [5], and to adopt refined processing routes [6] in order to develop tough ceramics with reliable strength. The major focus of the present work is however the strength characterization of brittle materials.

In a recent paper, Lu et al. [7] analyzed the fracture statistics of brittle materials using Weibull and normal distributions. They have considered the strength data of three different ceramic materials, i.e. silicon nitride (Si_3N_4), silicon carbide (SiC) and zinc oxide (ZnO). They used three-parameter

* Correspondingauthor. Tel.: +91 512 2597771; fax: +91 512 2597505.
E-mail address: bikram@iitk.ac.in (B. Basu).
[1] On leave from Department of Applied Mechanics, Indian Institute of Technology Delhi, India.

0272-8842/$34.00 © 2007 Elsevier Ltd and Techna Group S.r.l. All rights reserved.
doi:10.1016/j.ceramint.2007.10.003

Weibull, two-parameter Weibull and normal distributions to analyze these data. It is observed that based on the Akaike information criterion (AIC), two-parameter Weibull or normal distributions fit better than the three-parameter Weibull distribution. Although two-parameter Weibull distribution has been widely used in practice to model strength data, Lu et al. [7] questioned the uncritical use of Weibull distribution in general.

In the present work, we analyze the strength data, obtained in our previous work on monolithic ZrO_2 and ZrO_2–TiB_2 composites. Additionally, two more strength datasets, one for glass (unknown composition) and other for Si_3N_4 ceramics are selected from available literature. Such a selection of strength dataset will allow us to statistically analyze the strength property of a range of materials i.e. extremely brittle solid like glass to relatively tougher engineering ceramics, like Si_3N_4/ZrO_2-based materials. In our analysis, a much larger class of probabilistic models has been used. It is to be noted that the strength is always positive and therefore, it is reasonable to analyze the strength data using the probability distribution, which has support only on the positive real axis. Based on this simple idea we have attempted different two-parameter distributions namely, Weibull, gamma, log-normal and generalized exponential distributions. It should be mentioned here that all the above distributions have shape and scale parameters. As the name suggests the shape parameter of each distribution governs the shape of the respective density and distribution functions. For comparison purposes, we have also fitted normal distribution to all the datasets, although it does not have the shape parameter and it has the support on the whole real line.

2. Physics of the fracture of brittle solids

The variability in strength of ceramics is primarily due to the extreme sensitivity of the presence of cracks of different sizes. It can be noted that the Yield strength and the fracture/failure strength of polycrystalline metals is deterministic and is volume independent, when the characteristic microstructural feature (grain size) remained the same for the tested metallic samples. However, the fracture strength of a brittle material is, in particular, determined by the critical crack length according to the Griffith's theory [8]:

$$\sigma_f = \frac{K_{IC}}{\sqrt{\pi a}},$$

where σ_f the failure or fracture strength, K_{IC}, the critical stress intensity factor (a measure of fracture toughness) under mode-I (tensile) loading and 'a' the half of the critical or largest crack size.

For a given ceramic material the distribution of crack size, shape, and orientation differs from sample to sample. It is experimentally reported that the strength of ceramics varies unpredictably even if identical specimens are tested under identical loading conditions [4]. In particular, the mean strength, as determined from a multiplicity of similar tests depends on volume of material stressed, shape of test specimen and nature of loading. It is recognized that strength property needs to be analyzed using different probabilistic approaches, largely because of the fact that the probability of failure or fracture of a given ceramic sample critically depends on the presence of a potentially dangerous crack of size greater than a characteristic critical crack size [4]. Clearly, the probability of finding critical crack size is higher in larger volume test specimens and consequently, the brittle materials do not have any deterministic strength property. Since brittle materials exhibit volume dependent strength behavior, the mean strength decreases as the specimen volume increases. From the initial experimental observations, it was evident that a definite relationship should exist between the probability that a specimen will fracture and the stress to which it is subjected. Based on the above observations/predictions, Weibull [9] proposed a two parameter distribution function to characterize the strength of brittle materials. The generalized strength distribution law has the following expression: $F(\sigma) = 1 - e^{-(V/V_0)g(\sigma)}$, where $F(\sigma)$ is the probability of failure at a given stress level 'σ', V is the volume of the material tested, V_0 is the reference volume and $g(\sigma)$ is the Weibull strength distribution function: $g(\sigma) = (\sigma/\sigma_0)^m$, where m is the Weibull modulus and σ_0 is the reference strength for a given reference volume V_0. The characteristic strength distribution parameter, m, indicates the nature, severity and dispersion of flaws [2]. More clearly, a low m value indicates non-uniform distribution of highly variable crack length (broad strength distribution), while a high m value implicates uniform distribution of highly homogeneous flaws with narrower strength distribution. Typically, for structural ceramics, m varies between 3 and 12, depending on the processing conditions [1]. The Weibull distribution function, till to-date, is widely used to model or characterize the fracture strength of various brittle materials like Al_2O_3, Si_3N_4, etc. [2,10,11].

3. Experiments

As part of the present study, the analysis of four strength datasets is performed. The first two datasets i.e. dataset 1 and dataset 2 are the results of previous experimental work. In particular, dataset 1 refers to the strength data obtained with hot pressed ZrO_2 (2.5 mol% yttria-stabilized) – 30 vol% TiB_2 (TZP–TiB_2) composites; while dataset 2 is obtained during the strength measurement of hot pressed 2 mol% yttria-stabilized tetragonal zirconia (2Y-TZP) monolithic ceramic. Both the selected materials are fully dense (97% theoretical density). The details of the processing, microstructural characterization as well mechanical properties can be found elsewhere [12–15]. The selection of these particular grades of ZrO_2 materials is primarily because of the fact that recent research in optimizing the toughness of TZP-based materials revealed that both the selected 2Y-TZP monoliths and the TZP-TiB_2 composite exhibited best fracture toughness (2Y-TZP: 10.2±0.4 MPa $m^{1/2}$; TZP-TiB_2: 10.3 ±0.5 MPa $m^{1/2}$) of all the developed materials [13–15]. Therefore, detailed tribiological characterization as well as strength measurement was carried out on these optimized materials [14]. The microstructural characterization study using SEM and TEM revealed the homogeneous distribution of coarser

TiB_2 particles (average size $\sim 1\ \mu m$) in TZP matrix. The average ZrO_2 grain size in both monolith and composite is ~ 0.3–0.4 μm. Because of the use of highly pure commercial starting powders, the presence of any grain boundary crystalline/amorphous phase neither in monolith nor in composite was detected using high resolution TEM study [15].

The flexural strength of both ZrO_2 monolith and composite at room temperature was measured using a 3-point bending test configuration. The test specimens with typical dimension of 25.0 mm × 5.4 mm × 2.1 mm, were machined out of the hot pressed disks. The span width was 20 mm with a cross head speed of 0.1 mm/min. At least 15 identical specimens were tested for each material grade. The fracture surface observations using SEM predominantly indicated intergranular fracture in both ZrO_2 monolith and composites. Also detailed microscopy study indicated similarity in fracture origin for both the selected materials i.e. the critical surface flaw, located on the tensile face of the bend specimen.

Among the four selected datasets, the other two datasets are taken from literature. While dataset 3 is obtained using sintered Si_3N_4 materials [16], the dataset 4 is reported to be recorded from the brittle glass of unknown composition [17]. It can be mentioned here that Si_3N_4-based materials have been widely researched in the ceramics community for their potential high temperature applications, like engine components, etc. The details of the strength measurements and microstructural details of the selected Si_3N_4 materials can be found elsewhere [16]. In Ref. [17], the 3-point flexural strength measurement is reported for an unknown glass compositions. Typical bend bar dimension of glass sample was 3 mm × 4 mm × 40 mm with span length of 30 mm. The crosshead velocity was 0.5 mm/min. All the strength data i.e. datasets 1,2,3 and 4 are tabulated in Tables 1–4 respectively.

Table 1
The experimentally measured flexural strength data (dataset 1, in MPa units) as obtained with hot pressed ZrO_2–TiB_2 composites

Sample no.	Value
1	495.1
2	628.7
3	1179.4
4	1121.2
5	1028.7
6	871.1
7	1077.0
8	1350.0
9	1320.5
10	1327.4
11	1070.6
12	1342.7
13	1177.6
14	1226.2
15	1160.5
16	1257.8
17	1214.4
18	1136.5
19	853.9
20	1084.4
21	1052.8
22	1116.3

Table 2
The experimentally measured flexural strength data (dataset 2, in MPa units) as recorded with hot pressed ZrO_2 ceramic

Sample no.	Value
1	1269.8
2	1290.1
3	1372.3
4	1128.8
5	1243.6
6	1287.6
7	1288.1
8	1381.9
9	995.3
10	698.5
11	649.8
12	937.1
13	1381.9
14	1228.7
15	1362.5
16	893.5
17	690.3
18	545.0
19	691.0
20	810.4
21	539.3
22	785.9
23	682.0
24	676.0
25	419.2
26	450.0
27	423.6
28	488.4
29	353.7
30	619.6
31	631.419
32	648.203
33	527.212

Table 3
The strength data (Dataset 3, in MPa units) of Si_3N_4 ceramic, taken from Ref. [16]

Sample no.	Value
1	373.32
2	421.76
3	421.87
4	450.97
5	464.01
6	511.14
7	517.5
8	512.99
9	556.07
10	560
11	571.4
12	722.13
13	796.52
14	800.84
15	820.66
16	833.4
17	839.03
18	885.85

Table 4
The strength data (dataset 4, in MPa units) of glass, taken from Ref. [17]

Sample no.	Value
1	47.7
2	50.2
3	52.4
4	52.5
5	52.9
6	53.8
7	53.9
8	54.6
9	54.7
10	54.9
11	55.3
12	55.5
13	56.4
14	57.5
15	59.0
16	60.0
17	61.1
18	61.4
19	62.4
20	62.7
21	63.2
22	63.5
23	64.2
24	65.4
25	65.4
26	65.6
27	66.3
28	66.6
29	66.6
30	66.8
31	67.2
32	67.5
33	67.6
34	68.0
35	68.4
36	69.6
37	70.4
38	70.7
39	72.6
40	74.4

4. Different competing models

In this section we briefly describe different competing probabilistic models considered here and mention the estimation procedures of the unknown parameters from a given sample dataset $\{x_1, \ldots, x_n\}$.

4.1. Weibull distribution

The density function of the two-parameter Weibull distribution for $\alpha > 0$ and $\lambda > 0$ has the following form:

$$f_{\mathrm{WE}}(x;\alpha,\lambda) = \alpha\lambda^{\alpha}x^{\alpha-1}\,\mathrm{e}^{-(\lambda x)^{\alpha}}. \tag{1}$$

Here α and λ represent the shape and scale parameters, respectively. Therefore, the maximum likelihood estimators of α and λ can be obtained by maximizing the following log-likelihood function with respect to the unknown parameters;

$$L_{\mathrm{WE}}(\alpha,\lambda|x_1,\ldots,x_n) = n\ln\alpha + (n\alpha)\ln\lambda + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \lambda^{\alpha}\sum_{i=1}^{n}x_i^{\alpha}. \tag{2}$$

Note that if $(\hat{\alpha},\hat{\lambda})$ maximize (2) then

$$\hat{\lambda} = \left(\frac{n}{\sum_{i=1}^{n}x_i^{\hat{\alpha}}}\right)^{(1/\hat{\alpha})}, \tag{3}$$

and $\hat{\alpha}$ can be obtained by maximizing the profile log-likelihood of α as given below;

$$P_{\mathrm{WE}}(\alpha) = n\ln\alpha - n\ln\left(\sum_{i=1}^{n}x_i^{\alpha}\right) + (\alpha-1)\sum_{i=1}^{n}\ln x_i. \tag{4}$$

Since (4) is a unimodal function, the maximization of $P_{\mathrm{WE}}(\alpha)$ is not a difficult problem.

4.2. Gamma distribution

The two-parameter gamma distribution for $\alpha > 0$ and $\lambda > 0$ has the following density function;

$$f_{\mathrm{GA}}(x;\alpha,\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\,\mathrm{e}^{-\lambda x}. \tag{5}$$

Here also α, λ represent the shape and scale parameters, respectively, and $\Gamma(\alpha)$ is the incomplete gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}\,\mathrm{e}^{-x}\,\mathrm{d}x.$$

The maximum likelihood estimators of α and λ can be obtained by maximizing the log-likelihood function

$$L_{\mathrm{GA}}(\alpha,\lambda|x_1,\ldots,x_n) = n\alpha\ln\lambda - n\ln(\Gamma(\lambda)) + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n}x_i. \tag{6}$$

with respect to the unknown parameters. Therefore, if $\hat{\alpha}$ and $\hat{\lambda}$ are the maximum likelihood estimators of α and λ, respectively, then

$$\hat{\lambda} = \frac{\hat{\alpha}}{(\frac{1}{n})\sum_{i=1}^{n}x_i}, \tag{7}$$

moreover, the maximum likelihood estimator of α can be obtained by maximizing

$$P_{\mathrm{GA}}(\alpha) = \alpha n(\ln\alpha - 1) - n\ln(\Gamma(\alpha)) + \alpha\sum_{i=1}^{n}\ln x_i. \tag{8}$$

4.3. Log-normal distribution

The density function of the two-parameter log-normal distribution with scale parameter λ and shape parameter α is as follows;

$$f_{LN}(x;\alpha,\lambda)=\frac{1}{\sqrt{2\pi}x\alpha}\mathrm{e}^{-[(\ln x-\ln\lambda)^2/2\alpha^2]}. \quad (9)$$

The maximum likelihood estimators of the unknown parameters can be obtained by maximizing the log-likelihood function of the observed data

$$L_{LN}(\alpha,\lambda|x_1,\ldots,x_n)=-\sum_{i=1}^{n}\ln x_i-n\ln\alpha-\sum_{i=1}^{n}\frac{(\ln x-\ln\lambda)^2}{\alpha^2}. \quad (10)$$

Interestingly, unlike Weibull or gamma distributions, the maximum likelihood estimators of α and λ can be obtained explicitly and they are as follows;

$$\hat{\lambda}=\left(\Pi_{i=1}^{n}x_i\right)^{(1/n)} \quad \text{and} \quad \hat{\alpha}=\left[\frac{1}{n}\sum_{i=1}^{n}(\ln x_i-\ln\hat{\lambda})^2\right]^{1/2}. \quad (11)$$

4.4. Generalized exponential distribution

The two-parameter generalized exponential distribution has the density function

$$f_{GE}(x;\alpha,\lambda)=\alpha\lambda\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-1}. \quad (12)$$

Here $\alpha>0$ and $\lambda>0$ are the shape and scale parameters, respectively. Based on the observed data, the log-likelihood function can be written as

$$L_{GE}(\alpha,\lambda|x_1,\ldots,x_n)=n\ln\alpha+n\ln\lambda-\lambda\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\ln(1-\mathrm{e}^{-\lambda x_i}). \quad (13)$$

Therefore, $\hat{\alpha}$, the maximum likelihood estimator of α, can be written as

$$\hat{\alpha}=-\frac{n}{\sum_{i=1}^{n}\ln(1-\mathrm{e}^{-\lambda x_i})}, \quad (14)$$

and the maximum likelihood estimator of λ can be obtained by maximizing the following profile log-likelihood of λ,

$$P_{GE}(\lambda)=-n\ln\left(\sum_{i=1}^{n}\ln(1-\mathrm{e}^{-\lambda x_i})\right)+n\ln\lambda-\lambda\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\ln(1-\mathrm{e}^{-\lambda x_i}). \quad (15)$$

5. Different discrimination procedures

In this section we describe different available methods for choosing the best fitted model to a given dataset. For notational simplicity it is assumed that we have only two different classes, but the method can be easily understood for arbitrary number of classes also. Suppose there are two families, say, $\mathcal{F}=\{f(x;\theta);\theta\in\mathcal{R}^p\}$ and $\mathcal{G}=\{g(x;\gamma);\gamma\in\mathcal{R}^q\}$, the problem is to choose the correct family for a given dataset $\{x_1,\ldots,x_n\}$. The following methods can be used for model discrimination.

5.1. Maximum likelihood criterion

Cox [18] proposes to choose the model which yields the largest likelihood function. Therefore, Cox's procedure can be described as follows. Let

$$T(\hat{\theta},\hat{\gamma})=\sum_{i=1}^{n}\ln\left(\frac{f(x_i|\hat{\theta})}{g(x_i|\hat{\gamma})}\right), \quad (16)$$

where $\hat{\theta}$ and $\hat{\gamma}$ are maximum likelihood estimators of θ and γ, respectively. Choose the family $\mathcal{F}$ if $T>0$, otherwise choose $\mathcal{G}$. The statistic T is sometimes called the Cox's statistic. It is also observed [7] that when properly normalized, the statistic $\ln T$ should be asymptotically normally distributed. White [19] studied the regularity conditions needed for the asymptotic distribution to hold. Marshal et al. [20] use the likelihood ratio test and by extensive simulation study, they determine the probability of correct selection for different sample sizes. Recently [21,22], different researchers exploit the asymptotic property of T and determine the minimum sample size which is required for discriminating between different competing models.

In terms of T, the above selection procedure is related to a procedure for testing the hypothesis that the sample came from $\mathcal{F}$ versus that it came from $\mathcal{G}$. This testing problem treats the two families asymmetrically and so it is slightly different from the selection problem described above and it is not pursued here.

5.2. Minimum distance criterion

Among competing models, it is natural to choose a particular model for a given sample, which has the distribution function *closest* to the empirical distribution function of the data according to some distance measure between the two distribution functions. Note that the empirical distribution function of the given data $\{x_1,\ldots,x_n\}$ is given by

$$F_n(x)=\frac{\text{Number of } x_i\le x}{n}. \quad (17)$$

The distance between two distribution functions can be defined in several ways, but the most popular distance function between two distribution functions, say F and G, is known as the Kolmogorov distance and it can be described as follows;

$$D(F,G)=\sup_{-\infty<x<\infty}|F(x)-G(x)|. \quad (18)$$

To implement this procedure, a candidate from each parametric family that has the smallest Kolmogorov distance should be found and then the different best fitted distributions should be compared. Unfortunately, the first step of this procedure is difficult both from a theoretical and computational

Table 5
Estimated parameters, *K–S* distances, log-likelihood values and the fitted chi-square values for different distribution functions of dataset 1

Distribution	Shape	Scale	Chi-square	Kolmogorov	Log-likelihood
Weibull	7.1033	0.0851	5.905	0.1433 (0.7568)	−45.8863
Gamma	14.9999	1.3698	4.507	0.2573 (0.0905)	−50.8466
Log-normal	0.2395	0.0936	7.642	0.2555 (0.1132)	−51.8826
Gen. Exp.	38.0767	0.3758	7.062	0.2648 (0.0914)	−53.4365
Normal	10.9513 (mean)	0.4683 (variance)	4.479	0.1960 (0.3667)	−47.9054

point of view. Practically, from each parametric family the best member is chosen by maximum likelihood estimators rather than minimizing Kolmogorov distance. Then the family is chosen that provides the best fit to the empirical distribution in the sense of Kolmogorov distance.

5.3. Minimum chi-square criterion

This is most probably the oldest method which is being used for goodness of fit or for model discrimination. The basic idea of the minimum chi-square criterion is very simple. First divide the sample in k different groups and count the number of observations in each groups. If $f(x,\tilde{\theta})$ and $g(x,\tilde{\gamma})$ are the best fitted models from the families $\mathcal{F}$, and $\mathcal{G}$, respectively, then compute the expected number of observations in each group based on $f(x,\tilde{\theta})$ and $g(x,\tilde{\gamma})$. Suppose the observed frequencies in each group are $n_1,\ldots,n_k$, and the expected frequencies based on $f(x,\tilde{\theta})$ and $g(x,\tilde{\gamma})$ are $f_1,\ldots,f_k$ and $g_1,\ldots,g_k$, respectively, then the chi-square distance between $\{x_1,\ldots,x_n\}$ and $f(x,\tilde{\theta})$ is defined as

$$\chi^2_{\mathrm{f,data}} = \sum_{i=1}^{k} \frac{(n_i - f_i)^2}{f_i}. \tag{19}$$

Similarly, the chi-square distance between $\{x_1,\ldots,x_n\}$ and $g(x,\tilde{\gamma})$ is

$$\chi^2_{\mathrm{g,data}} = \sum_{i=1}^{k} \frac{(n_i - g_i)^2}{g_i}. \tag{20}$$

Now between the two families $\mathcal{F}$ and $\mathcal{G}$ choose family $\mathcal{F}$ if $\chi^2_{\mathrm{f,data}} < \chi^2_{\mathrm{g,data}}$ and choose family $\mathcal{G}$ otherwise. In this case also, like the previous one, from a given family the best model is chosen using the maximum likelihood estimators. Therefore, $\tilde{\theta}$ and $\tilde{\gamma}$ are chosen as $\hat{\theta}$ and $\hat{\gamma}$, respectively see Tables 5–8.

Table 6
Estimated parameters, *K–S* distances, log-likelihood values and the fitted chi-square values for different distribution functions of dataset 2

Distribution	Shape	Scale	Chi-square	Kolmogorov	Log-likelihood
Weibull	2.8045	0.1030	5.007	0.1873 (0.1971)	−85.9770
Gamma	6.3305	0.7358	2.223	0.1634 (0.3137)	−85.5922
Log-normal	0.4087	0.1260	1.040	0.1608 (0.3607)	−85.6416
Gen. Exp.	10.2548	0.3432	1.188	0.1617 (0.3538)	−85.6909
Normal	8.6032 (mean)	0.2956 (variance)	6.528	0.1989 (0.1467)	−85.6209

Table 7
Estimated parameters, *K–S* distances, log-likelihood values and the fitted chi-square values for different distribution functions of dataset 3

Distribution	Shape	Scale	Chi-square	Kolmogorov	Log-likelihood
Weibull	4.0414	0.1472	6.0954	0.2195 (0.3509)	−34.9123
Gamma	13.3289	2.1696	3.8959	0.1933 (0.4676)	−34.4537
Log-normal	0.1691	0.2760	3.2291	0.1929 (0.5145)	−34.3606
Gen. Exp.	45.0044	0.7167	2.7140	0.1944 (0.5041)	−34.3361
Normal	6.1441 (mean)	0.5916 (variance)	5.6046	0.2115 (0.3962)	−34.9905

Table 8
Estimated parameters, *K–S* distances, log-likelihood values and the fitted chi-square values for different distribution functions of dataset 4

Distribution	Shape	Scale	Chi-square	Kolmogorov	Log-likelihood
Weibull	2.7520	0.5318	3.9230	0.1389 (0.4234)	−40.6037
Gamma	4.7699	2.8517	6.6101	0.1508(0.3008)	−43.1471
Log-normal	0.4056	0.5118	10.3127	0.1640(0.2324)	−46.1947
Gen. Exp.	5.4811	1.3925	8.7322	0.1511(0.3200)	−44.5283
Normal	1.6723 (mean)	0.4537 (variance)	4.1914	0.1324 (0.4842)	−40.9528

6. Experimental results

For datasets 1, 2, 3 and 4, we have fitted different distributions and the estimated parameter values, chi-square values, Kolmogorov distances and the log-likelihood values are reported in Tables 5–8, respectively. For datasets 1–3, we have divided each data point by 100 and for dataset 4, we subtracted 45 and divided by 10. For each dataset the observed and expected values due to different fitted distributions are also reported in Tables 9–12, respectively. We also provide the empirical survival functions and the fitted survival functions for different distributions and for both the datasets in Figs. 1–4, respectively.

From Table 5(see also Fig. 1), it is clear that for dataset 1, Weibull is the best fitted model based on the maximum Likelihood criterion or the minimum Kolmogorov distance criterion followed by normal distribution. However, the chi-square value is not the minimum for dataset 1. Since it is well known that the chi-square value may not be that reliable, we accept that for dataset 1, Weibull is the best fitted model among different models considered here. Similar phenomenon is observed for dataset 4. For this set also it is observed that Weibull is the best fitted model in terms of all the criteria (see Table 8 and Fig. 4).

The picture is quite different for dataset 2 (see Table 6 and Fig. 2) and dataset 3 (see Table 7 and Fig. 3). Based on the log-likelihood values, Kolmogorov distance and also the chi-square values, Weibull is the worst fitted model. For dataset 2, apparently gamma and for dataset 3, log-normal are the best fitted models.

Another point that can be mentioned is that the fitted Weibull and normal distributions are closer to each other when compared to the fitted gamma, log-normal and generalized exponential distributions. Therefore, we can make two classes, one with Weibull and normal distributions and the other with gamma, log-normal and generalized exponential distributions. Suppose we take one representative distribution from each group, say Weibull and log-normal. Then based on the result

Table 9
Actual and expected number of observation at different intervals for different distribution functions of dataset 1

Interval	Observation	Weibull	Gamma	Log-normal	Gen. Exp.	Normal
<6	1	0.18	0.47	0.18	0.32	0.22
6–9	3	5.04	5.22	2.89	5.58	3.74
9–11	5	6.85	6.22	7.16	5.99	7.23
11–13	9	5.39	5.23	8.94	4.58	7.09
>13	4	4.54	4.86	2.83	5.52	3.71

Table 10
Actual and expected number of observation at different intervals for different distribution functions of dataset 2

Interval	Observation	Weibull	Gamma	Log-normal	Gen. Exp.	Normal
<4	1	2.63	1.92	1.55	1.65	2.86
4–6	7	4.90	5.87	6.61	6.49	4.42
6–8	10	7.00	8.09	8.61	8.57	6.88
8–13	11	15.03	13.59	12.49	12.59	15.64
>13	4	3.42	3.53	3.74	3.70	3.20

Table 11
Actual and expected number of observation at different intervals for different distribution functions of dataset 3

Interval	Observation	Weibull	Gamma	Log-normal	Gen. Exp.	Normal
<400	1	1.99	1.54	1.41	1.29	1.81
401–600	10	6.18	7.50	7.96	8.44	6.55
601–800	3	7.23	6.51	6.16	5.82	7.16
>801	4	2.60	2.45	2.46	2.44	2.45

Table 12
Actual and expected number of observation at different intervals for different distribution functions of dataset 4

Interval	Observation	Weibull	Gamma	Log-normal	Gen. Exp.	Normal
<0.75	3	3.06	3.32	3.51	3.17	3.42
0.75–1.25	10	8.03	9.73	10.92	10.19	7.20
1.25–1.75	6	11.29	10.93	10.30	10.32	11.22
1.75–2.25	12	9.84	7.89	6.70	7.11	10.34
2.25–2.75	9	5.45	4.54	3.84	4.13	5.63
>2.75	2	2.32	3.67	4.73	4.53	2.19

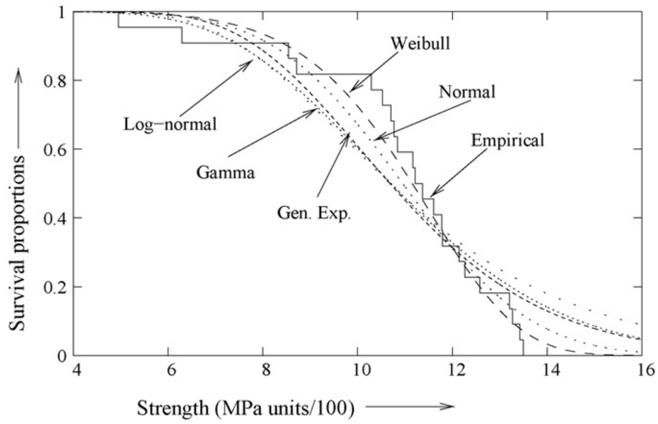

Fig. 1. Empirical survival function (bold line) and the fitted survival functions (dotted lines) for dataset 1 (ZrO_2–TiB_2 composite).

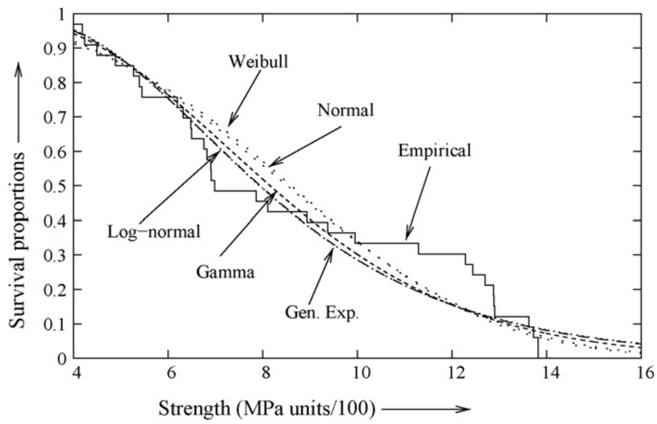

Fig. 2. Empirical survival function (bold line) and the fitted survival functions (dotted lines) for dataset 2 (ZrO_2 ceramic).

[21], it is possible to find the probability of correct selection in each case. In fact, the probability of correct selection for datasets 1, 2, 3 and 4 are approximately 78%, 82%, 77% and 85%, respectively. Therefore, they are quite high.

For dataset 1, it can be noted that shape parameter of the Weibull distribution is very high. It shows the symmetric nature of the data whereas dataset 2 is more skewed. Therefore, it is

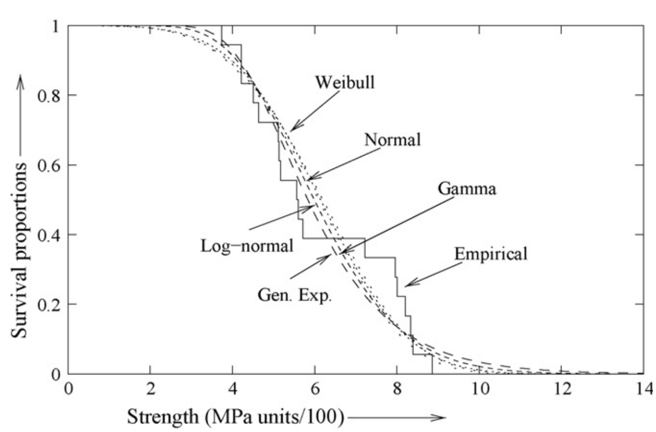

Fig. 3. Empirical survival function (bold line) and the fitted survival functions (dotted lines) for dataset 3 (Si_3N_4 ceramic).

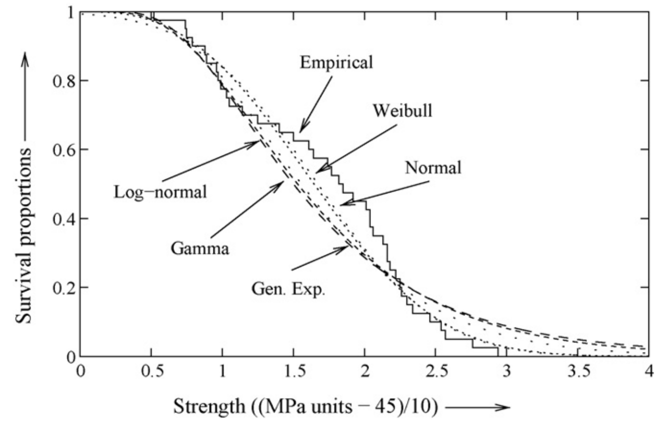

Fig. 4. Empirical survival function (bold line) and the fitted survival functions (dotted lines) for dataset 4 (glass).

clear that if the strength data are distributed symmetrically around its mean, then Weibull distribution may provide a good fit. However, if it is not, then there may be several good competitors. In our opinion, normal distribution should not be used in fitting strength data, because it may take negative values with high probability.

In view of the presented statistical analysis as well as that of Lu et al. [7], it is important to revisit the basic theory of Weibull, which links the statistical probability of fracture to the probability of finding a critical crack size in the tested sample. Further investigation should focus on rationalizing/justifying other strength distribution function from the perspective of the probabilistic theory of brittle fracture.

As a concluding note, the uncritical use of Weibull distribution must be avoided and therefore, the use of Weibull modulus as a strength reliability parameter can only be made after detailed analysis of strength data, as presented in this paper. Similar to the strength data, the grain size parameters, like mean grain size, grain size distribution width are equally important factors in determining critical material properties. In one of our earlier studies [22], the use of several statistical distribution functions, like normal, log-normal, Gumbel (Extreme value of type I) was made to evaluate the appropriate distribution function for microstructural description of sintered ceramics, like ZrO_2. It was concluded from that study that Gumbel distribution describes much better (statistically) the grain size distribution. However, in many studies, the uncritical use of Gaussian or normal distribution were made to find out grain size distribution parameters for several metals/ceramic materials. The above discussions evidently places the importance of detailed statistical analysis in evaluating the properties of materials i.e. in a larger scale, in the field of material science.

7. Conclusions

In the present work, we have considered several statistical distribution functions with an aim to critically analyze the strength data of brittle materials, like ceramics. Other than Weibull and normal, several two-parameter distributions, like

Gamma, log-normal and generalized exponential distributions were used. The experimentally measured strength data obtained with hot pressed dense ceramics, like monolithic ZrO_2, ZrO_2–TiB_2 composites as well as literature strength data of Si_3N_4 ceramic and glass were used to validate the statistical analysis. It is observed that the fitted Weibull and normal distributions behave quite similarly, whereas the fitted gamma, log-normal and generalized exponential distributions are of similar nature. Based on the limited set of strength data and using several statistical criteria, like minimum chi-square, minimum Kolmogorov distance and maximum log-likelihood value, the gamma or log-normal distribution function appears to be more appropriate statistical distribution function in some investigated cases. Another important result has been that the probability of correct selection for datasets 1, 2, 3 and 4 are approximately 78%, 82%, 77% and 85%, respectively, which are quite high.

The implication of our study is important and that is the strength property of brittle ceramics should be characterized using various statistical criteria and different distribution functions, as adopted in the present work.

Acknowledgment

The authors would like to thank Mr. Damodar Tiwari for his constructive suggestions to prepare this manuscript.

References

[1] Y.M. Chiang, D. Birnie, W.D. Kingery, Physical Ceramics: Principles for Ceramic Science and Engineering, John Wiley & Sons Inc., New York, 1997.
[2] R.W. Davidge, Mechanical Behavior of Ceramics, Cambridge University Press, Cambridge, UK, 1979.
[3] J.B. Wachtman, Mechanical Properties of Ceramics, John Wiley & Sons Inc., New York, 1996.
[4] B. Lawn, Fracture of Brittle Solids, second ed., Cambridge University Press, Cambridge, UK, 1998.
[5] B. Basu, Toughening of Y-stabilized tetragonal zirconia ceramics, Int. Mater. Rev. 50 (4) (2005) 239–256.
[6] F.F. Lange, Powder processing science and technology for increased reliability, J. Am. Ceram. Soc. 72 (1998) 3–15.
[7] C. Lu, R. Danzer, D. Fishcer, Fracture statistics of brittle materials: Weibull or normal distribution, Phys. Rev. E 65 (2002) 067102.
[8] A. Griffith, Philos. Trans. R. Soc. A 221 (1920) 163.
[9] W. Weibull, A statistical Distribution Function of wide Applicability, J. Appl. Mech. 18 (1951) 293–305.
[10] Z.P. Bazant, Probability distribution of energetic-statistical size effect in quasibrittle fracture, Prob. Eng. Mech. 19 (2004) 307–391.
[11] R. Danzer, T. Lube, P. Supancic, Monte Carlo simulations of strength distributions of brittle materials - type of distribution, specimen and sample size, Z. Metallkd. 92 (2001) 773–783.
[12] Basu, B., Zirconia-Titanium Boride Composites for Tribiological Applications, Ph.D. thesis, Katholieke Universiteit Leuven, Belgium, 2001.
[13] B. Basu, J. Vleugels, O. Van Der Biest, Toughness tailoring of yttria-doped zirconia ceramics, Mater. Sci. Eng. A 380 (2004) 215–221.
[14] B. Basu, J. Vleugels, O. Van Der Biest, Microstructure-Toughness-Wear relationship of tetragonal zirconia ceramics, J. Eur. Ceram. Soc. 24 (7) (2004) 2031–2040.
[15] Basu, B., Vleugels, J. and Van Der Biest, O., Processing and Mechanical properties of ZrO_2–TiB_2 composites, J. Eur. Ceram. Soc. 25 (2005) 3629–3637.
[16] G.D. Quinn, R. Morrell, Design data for engineering ceramics - a review of the flexure test, J. Am. Ceram. Soc. 74 (1991) 2037–2066.
[17] Y. Lu, L. Cheng, L. Zhang, D. Yan, C. You, Optimization of sample number for Weibull functions of brittle materials strength, Ceram. Int. 27 (2001) 239–241.
[18] D.R. Cox, Further results on tests of separate families of hypothesis, J. R. Stat. Soc. Ser. B 24 (1962) 406–424.
[19] A.W. Marshal, J.C. Meza, I. Olkin, Can data recognize its parent distribution? J. Comput. Graph. Stat. 10 (2001) 555–580.
[20] D. Kundu, A. Manglick, Discriminating between the Weibull and Log-normal distributions, Naval Res. Log. 51 (2004) 893–905.
[21] D. Kundu, R.D. Gupta, A. Manglick, Discriminating Between the Log-Normal and the Generalized Exponential Distributions, J. Stat. Plann. Inference 127 (2005) 213–227.
[22] D. Dierickx, B. Basu, J. Vleugels, O. Van Der Biest, Statistical extreme value modelling of particle size distributions: experimental grain size distribution type estimation and parameterisation of sintered zirconia, Mater. Character. 45 (2000) 61–70.

Bikramjit Basu Dr. Basu obtained his Masters (Metallurgy) in 1997 from Indian Institute of Science, Bangalore. Subsequently, he pursued his doctoral research in the area of ceramics at an old and reputed European school i.e. Katholieke Universiteit Leuven (KUL), Belgium and obtained his PhD degree in March, 2001. As an active member of European Commission sponsored BRITE-EURAM project, his research involved two aspects, (a) toughness optimization and understanding toughening mechanisms of Y-TZP composites and (b) establish microstructure-wear resistance relationship of ZrO2/TiB2 composites under various operating conditions. After a brief Post-Doctoral research stay at University of California, he joined Indian Institute of Technology Kanpur, India as Assistant Professor in November 2001. He held visiting positions at University of Warwick, Seoul National University and UPC, Barcelona.

In the span of more than a decade of his research activities in the broad area of Ceramic Science and Technology, Dr. Basu's primary research focus was to establish processing-structure-property relationship for ceramic composites, in particular Nanostructured ceramics and Biomaterials. In India, Dr. Basu established a strong ceramic research programme at IIT Kanpur with the total funding of *USD 500,000* from all major Governmental funding agencies. He has authored/co-authored more than *100 research papers* (including four invited review papers and one book chapter) in peer-reviewed International journals. He has delivered *40 invited lectures* both nationally and internationally, including in USA, UK, Canada, France, Belgium, Spain, and Korea.

In the area of *structural ceramics*, his research revealed that the addition of $MoSi_2$ or $TiSi_2$ could reduce the sintering temperature to 1700 ° C or lower. Using thermodynamic analysis and high resolution Electron Microscopy study, his research could establish the underlying densification mechanisms of new ceramic composite systems, including TiB_2–$MoSi_2$ and WC–ZrO_2 nanoceramic composite. Extremely high hardness of more than 22 GPa was measured in these newly developed ceramics. Relatively tougher ceramics, based on Ti_3SiC_2 and S-phase SiAlON also form a part of the past research activities of Dr. Basu's group. Using novel processing technique i.e. Spark plasma sintering, he has been able to develop various *Nanoceramic materials*, e.g. Y–TZP, ZrO_2–ZrB_2, and WC–Co–ZrO_2. At present, his primary research activities are in the area of designing *biomaterials* for hard tissue replacement/components with a focus to study dissociation mechanisms of hydroxyapatite (HA) in HA–Ti and HA–Mullite systems. The toughness improvement has been the major driver for his research on development of new Bioceramic composites. In the area of Tribology, he designed high speed cryo-tribometer to evaluate and understand the friction and wear mechanisms of ceramic bearings for space applications. On the fundamental aspect, his research focuses on correlating the *wear micromechanisms* with the material properties for a range of ceramic systems, including mixed carbide cermets: (W,Ti)C–Co, Y–TZP.

In recognition of his outstanding contribution in the field of Materials Science, in particular Ceramic Science, Dr. Basu is awarded '*Young Scientist Awad-2003*' by Indian Ceramics Society, '*Young Metallurgist Award - 2004*' by Indian

Institute of Metals, '*Young Engineer Award-2004*' by Indian National Academy of Engineering (INAE) and Indian National Science Academy (*INSA*) medal for '*Young Scientist*' in Engineering Sciences, 2005. Dr. Basu has been invited as a foreign member on the editorial board of *Journal of Korean Ceramic Society.* Dr. Basu is the Guest Editor of two topical issues of *International Journal of Applied Ceramic Technology* (*IJACT*) in the area of Bioceramics and Nanoceramic composites. In addition, he is also acting as Guest editor for topical issue for *Journal of Biomedical Materials Research* (*JBMR-B*) as well as for *Journal of Materials Science*. He is also the principal editor of the book "*Advanced Biomaterials: Fundamentals, Processing and Applications*", to be published by John Wiley & Sons Inc.

As an active member of the Engineering Science Division of American Ceramic Society, Dr. Basu was involved in chairing technical sessions and participated in the Executive Committee meeting of IJACT journal at the 31st International Cocoa Beach Conference. He is one of the organizers of Symposium on Next generation Bioceramics at the 32nd Cocoa beach conference. Dr. Basu was the *key organizer* of the International Conference on Design of Biomaterials (*BIND-06*). Recently, he organized International Workshop on Nanoceramics and Nanocomposites and he is also co-organiser of METALLO-07 conference. Till todate, he has supervised one PhD student and more than 15 masters students. At present, he is guiding four PhD students at IIT-Kanpur, India. As a faculty member at IIT-Kanpur, he has been rated as the best instructor for several undergraduate and postgraduate courses, like 'Materials for Biomdeical applications', 'Nanomaterials: Processing and Properties', 'Selection and Designing with Engineering Materials', 'Nature and Properties of Materials' and "Tribology of Materials".