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Statistical Determination of Surface Flaw Distributions in Brittle Materials

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Abstract

A computational method is presented for determining the distribution of surface flaws in brittle materials from a series of fracture tests. The number of cracks per unit area with sizes within a range c *to* c + Δc *can be determined by the solution of a system of simultaneous linear equations. As long as the stress intensity factor for the cracks causing fracture can be calculated then results can be obtained from any loading method; the method therefore assumes that the shape of the cracks causing fracture is known. The only data inputs are the fracture loads themselves. Accurate estimates of the flaw density can only be obtained if proper consideration is given to the variation of the stress intensity factor with the angle between the crack-face normal and the tensile stress. Results are shown for computer simulations of 3-point bend testing.*

1 Introduction

The strength of ceramic materials is, unlike that of most metals, inherently variable; fracture occurs over a range of stresses as a result of a distribution of flaw sizes and positions. The most common way of describing the distribution of flaws is that due to Weibull.[1] The Weibull failure probability is defined in its simplest form as:[2]

$$F = 1 - \exp\left[-(\sigma_I/\sigma_o)^m\right] \qquad (1a)$$

where σ_I is the inert strength of the material, m is the Weibull modulus and σ_o a normalizing parameter. In the case where the stress varies with position in the surface of the body, as for instance in a three-point bend test, then the Weibull failure probability becomes:

$$F = 1 - \exp\left[-\int\left(\frac{\sigma_I}{\sigma_o}\right)^m \frac{dA}{A_o}\right] \qquad (1b)$$

σ_o and A_o are normalizing parameters. For equation (1(a)), by forming the cumulative probability of failure, a plot of $\ell n\{\ell n[1/(1-F)]\}$ against $\ell n(\sigma_I)$ should give a straight line of slope m and intercept $-m\ell n(\sigma_o)$. However, it is well known that simple Weibull statistics are not a good description of failure probabilities, particularly if multiple flaw populations are present.[3] Furthermore, this approach gives no direct information about the distribution of the flaws themselves, in terms of the number of flaws in a given size range per unit area, $n(c)\Delta c$.

Argon[4,5] proposed using the Hertzian indentation test as a way of determining surface flaw densities in glass. In this test a hard sphere is pressed against the surface of a solid until fracture occurs (the formation of a ring-crack or ring-cone crack). Argon measured *just* the fracture loads and determined $n(c)\Delta c$ by numerical inversion of Laplace transforms. Wilshaw[6] and Poloniecki and Wilshaw[7], again using Hertzian indentation, presented a different method of analysis. They were motivated by the fact that Argon, 'by analysing only the fracture forces, made the problem unnecessarily complicated'. By measuring the fracture loads *and* the radii of ring-cracks seen on the glass surface *after* fracture, the sizes of the small precursor surface flaws could be determined. For a given test and crack size, the 'searched area' could be defined: 'that area of surface within which, had there been a crack of size c, it would have caused fracture during loading up to a load P'. Thus the searched area, $A(P,c)$, is that area of surface within which $K_I > K_{IC}$ for a crack of size c during loading up to P. The flaw density is simply that number of cracks found within a given size range (c to $c+\Delta c$) divided by the total area searched for flaws of that size (c) in all the tests. This method is not restricted to Hertzian indentation, and is in principle applicable to any test method. It is thus an extremely powerful method of determining flaw densities although it suffers from the disadvantage that it is necessary to find the position from which failure started.

Matthews, McLintock and Shack[8] derived a different method for determining $n(c)\Delta c$ from *just* the fracture loads measured in 3-point bend tests or Hertzian indentation tests; this method relied on the analytical inversion of an integral equation, which is possible only in limited cases. Evans and Jones[9] extended the work of Matthews *et al.* to include straight-forward tension, the expanding-ring tensile test and biaxial flexure, but again analytical inversion of the fundamental integral equation was crucial to the success of the method.

Alternatively, various analytical forms for the flaw distribution have been proposed. That due to Jayatilaka and Trustrum[10] is:

$$f(a) = \frac{c^{n-1}}{(n-2)!} a^{-n} e^{-c/a} \tag{2a}$$

where $f(a)$ is the probability density of cracks of semi-length a; c is a scaling parameter (*not* the crack size in this case) and n is a measure of the rate at which the flaw density approaches zero. In fact, $2n\text{-}2=m$, where m is the Weibull modulus. That due to Batdorf and Crose[11] and Batdorf and Heinisch[12] is:

$$N_s(\sigma) = k\sigma^m \tag{2b}$$

where $N_s(\sigma)$ is the surface crack density function, k is the crack density coefficient and m is the Weibull modulus. Recently, Santhanam and Shaw[13] determined flaw density distributions in ceramic tubes fractured by the crack-off process: by using a weakest-link approach (see below) combined with a flaw distribution function of the form of eqn (2a) (replacing the factorial function with the Γ function to allow for non-integral n) a best-fit of the function $f(a)$ to the fracture data was performed, leading to values for c and n.

In summary, there are three distinct approaches for determining $n(c)\Delta c$, the number of flaws per unit area in a size range Δc centred on c

(1) One can measure a fracture load *and* a fracture position (in which case the size of the crack that caused fracture can be determined); one then uses Wilshaw's[6,7] concept of 'searched area' to determine the flaw density.
(2) Alternatively, one can follow the approach of Matthews *et al.*[8] (or Evans and Jones)[9] where one measures *just* the fracture loads, and then use analytical inversion of an integral equation to determine $n(c)\Delta c$.
(3) Finally, one can follow Santhanam and Shaw's[13] approach, by assuming a functional form for $n(c)\Delta c$ and obtaining the best values of the adjustable parameters in the expression for $f(a)$.

There are problems with all three approaches. For method (1), in the case of Hertzian indentation it can be extremely difficult to find the ring-cracks formed after fracture, especially on surfaces that are not highly polished. One usually has to resort to the use of chemical etches to bring out the cracks, which in the case of ceramic materials such as alumina means using molten KOH at 360°C. This is not convenient. The second problem with method (1) is that it is not possible to automate it. In principle this means that method (2) is preferable. Unfortunately, the calculations necessary to obtain $n(c)\Delta c$ require (for the analytical inversion of the integral equation) that the form of the stress-field experienced by the cracks is simple, so that the method is not of general application. For Hertzian indentation Argon,[4,5] Wilshaw[6,7] and Matthews *et al.*[8] all assumed that the only stress acting on the crack was the surface radial stress; the steep stress gradients that exist in the Hertzian field were ignored. As pointed out by Lawn,[2] the issue of stress gradients has been largely overlooked by flaw statistics analysts. Finally, in method (3) the problem lies in *assuming* a probability distribution of a particular form. While the form given above, eqn (2) is analytically straightforward, it forces the density of small cracks to be small. For many purposes, (e.g. strength of ceramics), knowledge of the density of small cracks may be irrelevant, but if one is investigating, for example, polishing processes then information about these cracks may be important. Secondly, by forcing the small-crack density to be small, one runs the risk of skewing the distribution at the large crack end.

In the method described below, these problems are overcome. The determination of $n(c)\Delta c$ is accomplished by knowledge of just the fracture loads in a series of tests; no prior assumptions are made about the form of $n(c)\Delta c$, and any loading system can be catered for, as long as K_I can be calculated. Thus the only assumption made here is one about the *shape* of the cracks causing fracture. Finally, most previous methods (with the exception of Santhanam and Shaw)[13] of estimating $n(c)\Delta c$ have ignored the effect of the variation in K_I with the angle between the crack-face normal and the tensile stress axis. It will be shown that this has a pronounced effect on the values of $n(c)\Delta c$ determined.

2 Weakest-link Statistics

Following Sigl[14] and Danzer[15] we can write the cumulative probability of failure of a body during loading to a load P, $F(P)$, in terms of the *cumulative number of critical flaws* in the surface of the body, $N(P)$:

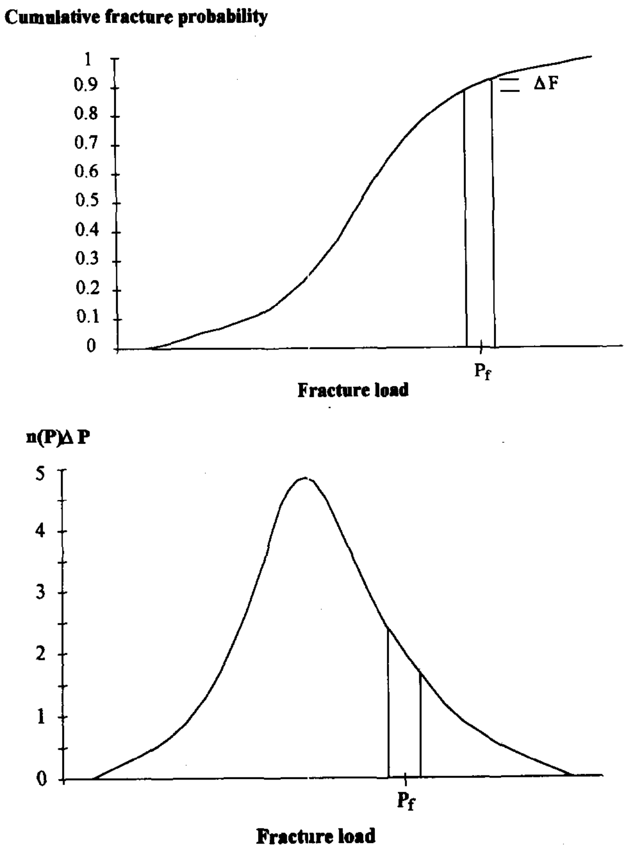

Fig. 1. Generic plots of: (a) cumulative probability of fracture $F(P)$, (b) number of fractures within a given load range $n(P)\Delta P$, both as a function of P.

$$F(P) = 1 - \exp(-N(P)) \quad (3)$$

Figure 1(a) shows a generic plot of $F(P)$, the cumulative probability of failure versus fracture load; Fig. 1(b) is the differential of this, $n(P)\Delta P$, the number of fracture loads in the range P to $P+\Delta P$. We define the number of flaws in a given size range per unit area, $n(c)\Delta c$: it is this quantity that is to be determined. Then, the number of flaws in the size range c to $c+\Delta c$ which are critical when the body is subjected to a load P is:

$$N'(c)\Delta c = A(P, c)n(c)\Delta c \quad (4)$$

where, following Wilshaw,[6] $A(P, c)$ is the *area searched* for flaws of size c during loading to P i.e. that area of material within which $K_I > K_{IC}$ for flaws of size c during loading to P. It is this quantity $A(P, c)$ that can be calculated for any loading system, assuming that the crack shape is known. For a distribution of crack sizes, we finally obtain:

$$F(P) = 1 - \exp\left[-\sum_j A(P, c_j)n(c_j)\Delta c\right] \quad (5)$$

A similar equation can obviously be derived for the three-dimensional case by replacing $A(P, c)$ by $V(P, c)$, the *volume* searched for flaws of size c during loading to P and by re-defining $n(c)\Delta c$ as the number of flaws in a given size range per unit volume.

Equation (5) can be re-cast in the form of a matrix equation:

$$\mathbf{b} = \mathbf{A}\mathbf{x} \quad (6)$$

b is a column vector with, say, m known components $b_i = -\ln[1-F(P_i)]$, **A** is an mxn matrix, whose known components are the searched areas, $A_{ij} = A(P_i, c_j)$ and **x** is a column vector with the n *unknown* components, $x_j = n(c_j)\Delta c$. **A** has $i = 1, m$ rows, corresponding to m points on the cumulative probability curve, $F(P_i)$, and $j = 1, n$ columns corresponding to those n crack sizes whose areal densities are to be determined. Strictly, m and n can be chosen arbitrarily, although as will be shown there are certain restrictions on the values that should be used in practice. We give below some examples of the utility of eqn (6).

3 Calculation of Searched Areas

In what follows we assume that the flaws controlling fracture are semi-circular surface-breaking cracks oriented normal to the tensile stress, σ. The stress intensity factor experienced by such a crack, depth c, is:

$$K_I = Y\sigma\sqrt{\pi c} \quad (7)$$

where Y is a shape factor, ~0·713.

3.1 Uniform tension

Figure 2 shows a rectangular parallelepiped pulled with a load P so that the tensile stress $\sigma = P/w^2$ where w^2 is the cross-sectional area of the block. Defining α_1 as:

$$\alpha_1 = \frac{w^2 K_{IC}}{Y\sigma\sqrt{\pi}} \quad (8a)$$

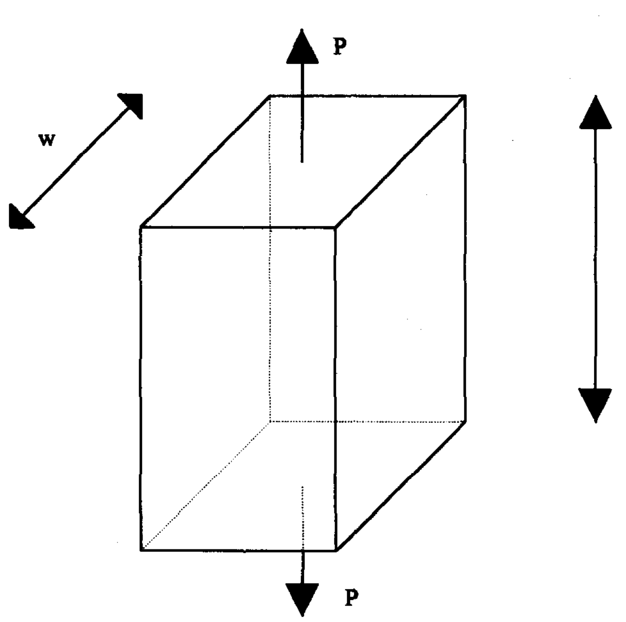

Fig. 2. Parallelepiped pulled by a uniform tension.

then, as long as $P_i \geq \alpha_1/\sqrt{c_j}$, the searched area is:

$$A(P_i, c_j) = 4w\ell \quad (8b)$$

Note that the area searched *during* loading up to P is the same as the area searched *at* a load P. It is perfectly possible to choose $m = n = 3$ (say), and also possible to choose the values of P_i and c_j such that eqn (6) takes the form:

$$\begin{pmatrix} 0 & 0 & A \\ 0 & A & A \\ A & A & A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (9a)$$

where the components of **b** are ranked in order of increasing load, and the components of **x** are ranked in order of increasing crack size. The solution is given by:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (b_3 - b_2)/A \\ (b_2 - b_1)/A \\ b_1/A \end{pmatrix} \quad (9b)$$

and it is clear that the density of the *largest* cracks, x_3, is controlled by the probability of fracture at the *lowest* load, b_1 as would be expected. Similarly, it is perfectly possible to set $m = n = 4$ and choose the values of P_4 and c_4 such that the matrix **A** has the form:

$$\begin{pmatrix} 0 & 0 & A & A \\ 0 & A & A & A \\ A & A & A & A \\ A & A & A & A \end{pmatrix} \quad (9c)$$

(This could be accomplished by choosing P_4 to be much bigger than P_3 but choosing c_4 to be only slightly larger than c_3). It is immediately clear that **A** is rank-deficient and that no unique solution for **x** exists. Surprisingly, it turns out that if $m = n$ then this is the normal situation; obviously if $m > n$, (i.e. one has more equations than unknowns) then there is never any unique solution for **x**. The method of solution of eqn (6) will be discussed in Section (5).

We now show that the matrix method of solution gives the same answers as those obtained by more conventional methods, at least in the simple case of uniform tension applied to a block. Suppose that a total of N_T fracture tests are performed and N_T fracture loads measured. From eqn (7) one can therefore derive N_T crack sizes. Let $n(P_f)\Delta P$ be the number of fracture loads in a load range ΔP, centred on P_f, this corresponds to the same number of cracks in a size range Δc, centred on c. The area searched for cracks in this size range is the product of the number of tests where the fracture load was greater than or equal to P_f (N_{crit}) with the area searched in each test (A), i.e. $N_{crit}A$. From Figs 1(a) and 1(b) we see that the fraction of tests with $P \geq P_f$ is $1\text{-}F(P_f)$. So, the total searched area is $N_{crit}A = N_T[1\text{-}F(P_f)]A$. But, also from Figs 1(a) and 1(b) the number of fracture loads that found cracks in this range is $N_T[\Delta F(P_f)]$, so that the crack density is:

$$n(c)\Delta c = \frac{\Delta F(P_f)}{[1 - F(P_f)]A} \quad (10)$$

By the matrix method, we can let $m = n$, and we can also choose the values of P and c such that A is triangular, i.e. all the upper-left components of A are 0. In this case, by analogy with eqn (9b) we see that the solution for the jth crack density is:

$$An(c_j)\Delta c = (b_{n-j+1} - b_{n-j}) = -\ell n\left[\frac{1 - F(P_{n-j+1})}{1 - F(P_{n-j})}\right] \quad (11a)$$

where P_{n-j} is the fracture load corresponding to a crack of size c_j. Letting n become large:

$$An(c)\Delta c = -\ell n\left[\frac{1 - F(P+\Delta P)}{1 - F(P)}\right] \quad (11b)$$

or, if $F(P + \Delta P) = F(P) + \Delta F(P)$, then:

$$n(c)\Delta c \approx \frac{\Delta F(P)}{[1 - F(P)]A} \quad (11c)$$

as before, eqn (10).

3.2 3-point bending

Referring to Fig. 3, we see that the tensile stress in the lower surface of the bar is given by:

$$\sigma(x) = \frac{3P(L-x)}{bh^2} \quad (12a)$$

Again, assuming that the only flaws present are semi-circular flaws oriented perpendicular to the tensile stress then the stress intensity factor for a flaw of depth c is as in eqn (7). For a load P_i and a flaw depth c_j the maximum distance that a flaw can be found from the centre of the tensile face is x_m:

$$x_m = L - \alpha_2/(P_i\sqrt{c_j}) \quad \alpha_2 = \frac{K_{IC}bh^2}{3Y\sqrt{\pi}} \quad (12b)$$

For $P_i \geq \alpha_2/(L\sqrt{c_j})$ the area searched $A(P_i, c_j)$ is:

$$A(P_1, c_j) = \int_0^{x_m} 2b\,dx = 2b\left[L - \frac{\alpha_2}{P_i\sqrt{c_j}}\right] \quad (12c)$$

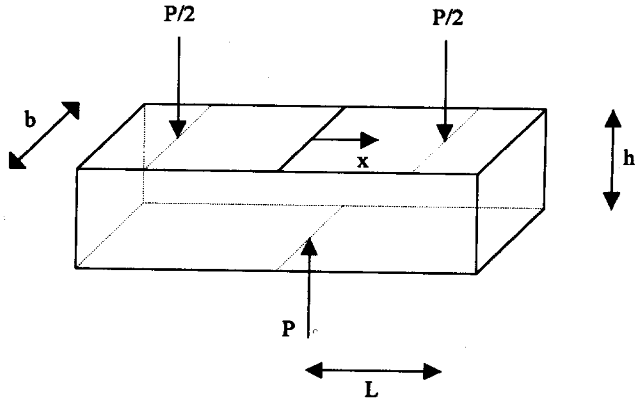

Fig. 3. Three-point bend test.

Note again that the area searched *during* loading up to P is the same as the area searched *at* a load P. In Appendix 1 we show that if eqn (6) is converted to its integral form and then combined with the expression for $A(P_i, c_j)$ from eqn (12c) one obtains the same expression for the flaw density as that originally derived by Matthews *et al.*[8]

3.3 Hertzian indentation

Finally, we consider an example where the area searched during loading to P is not the same as the area searched at a load P, the example of Hertzian indentation. If a sphere, radius R, elastic constants ν_s, E_s, is pressed with a load P into the flat surface of a substrate, elastic constants ν, E, then the two solids touch over a circular contact patch, radius a where:

$$a = \left(\frac{3RP}{4E^*}\right)^{1/3} \qquad \frac{1}{E^*} = \frac{1-\nu^2}{E} + \frac{1-\nu_s^2}{E_s} \quad (13a)$$

The stress state set up outside the contact patch *in the surface of the substrate* is one of radial tension, σ_{rr}, and hoop compression, $\sigma_{\theta\theta}$:

$$\sigma_{rr} = -\sigma_{\theta\theta} = \frac{(1-2v)}{2\pi}\frac{P}{r^2} \qquad r \geq a \qquad (13b)$$

With K_I given by eqn (7) then for a load P_i and a flaw depth c_j the maximum distance that a flaw can be found from the centre of the indentation is r_m:

$$r_m = \left(\frac{P\sqrt{c}}{\alpha_3}\right)^{1/2} \qquad \alpha_3 = \frac{2\sqrt{\pi}K_{IC}}{Y(1-2v)} \qquad (13c)$$

For $P_i \geq a^2\alpha_3/\sqrt{c_j}$ then the searched area *at* a load P_i is:

$$A(P_1, c_j) = \int_a^{r_m} 2\pi r\,dr = \pi(r_m^2 - a^2) \qquad (13d)$$

However, as originally shown by Wilshaw[6] $K_I > K_{IC}$ for loads lower than P_i. The minimum load at which $K_I = K_{IC}$, for a crack size c, is P_{min}. The crack must be situated at the position of maximum tensile stress, i.e. at the corresponding contact radius, a_{min}:

$$P_{min} = \left(\frac{3R\alpha_3}{4E^*}\right)^2 \frac{\alpha_3}{c^{3/2}} \qquad a_{min} = \frac{3R\alpha_3}{4E^* c^{3/2}} \quad (13e)$$

The total area searched in loading up to P_i is therefore:

$$A(P_i, c_j) = \pi(r_m^2 - a_{min}^2) \qquad (13f)$$

The variation of the maximum value of K_I with distance from the centre of the contact during loading is of importance for Section (4). At a maximum load P (and a corresponding contact radius a), for a given position $r (r \geq a)$ then the maximum K_I at that position is given by:

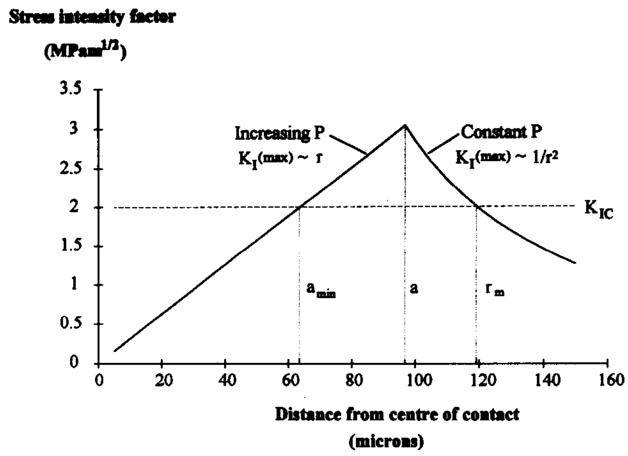

Fig. 4. Hertzian indentation: plot of maximum value of K_I as a function of distance from the centre of the contact using eqns (13g) and (13h), for the particular case $Y = 1{\cdot}12$, $\nu = 0{\cdot}24$, $E = 390$ GPa, $R = 2{\cdot}5$ mm, $K_{IC} = 2$ MPam$^{1/2}$, $P = 100$ N and $c = 3\mu$m.

$$K_I(\max) = \frac{Y(1-2v)}{2\pi}\frac{P}{r^2}\sqrt{\pi c} \qquad r \geq a \qquad (13g)$$

However, for loads lower than P (but greater than P_{min}), the maximum K_I occurs at $r = a$. Therefore, replacing P by $4E^*r^3/3R$ in eqn (13g) we obtain:

$$K_I(\max) = \frac{Y(1-2v)}{2\pi}\frac{4E^*}{3R} r\sqrt{\pi c} \quad a_{min} \leq r \leq a \, (13h)$$

Figure 4 shows a plot of K_I(max) as a function of distance from the centre of the contact for the particular case $Y = 1{\cdot}12$, $\nu = 0{\cdot}24$, $E = 390$ GPa, $R = 2{\cdot}5$ mm, $K_{IC} = 2$ MPam$^{1/2}$, $P = 100$ N and $c = 3$ μm.

4 Angular Variation

We now consider how the expressions for $A(P_i, c_j)$ are modified if the cracks are no longer oriented perpendicular to the tensile stress, but are skewed round at an angle θ. (The cracks are still assumed to be perpendicular to the specimen surface). We now replace the elemental searched area ΔA by the *angularly-weighted* elemental searched area: $\Delta A' = \phi(\Delta A)$ where ϕ is defined as the ratio of the angular range within which $K_I = K_{IC}$ (for a given P and c), $2\phi_c$, to the angular range within which cracks can possibly be found, ϕ_0. Assuming that the angular distribution of the cracks is uniform then $\phi_0 = \pi$; this assumption will be acceptable for normal polishing and lapping procedures but may not be true if the ceramic surface has been machined in one direction only.

For Sections (4.1) and (4.2) below, eqn (7) is replaced by:

$$K_I = Y\sigma\sqrt{\pi c}\cos^2(\theta) \qquad (14)$$

We make the assumption that mode I fracture still predominates, i.e. that the skewing of the crack does not give rise to significant mode II or mode III stress intensity factors.

4.1 Uniform tension

With α_1 as in eqn (8a) we have:

$$\phi_c = \cos^{-1}\left[\left(\frac{\alpha_1}{P\sqrt{c}}\right)^{1/2}\right] \quad (15a)$$

With $P_i \geq \alpha_1/\sqrt{c_j}$ the searched area becomes:

$$A'(P_i, c_j) = \frac{8w\ell}{\pi}\cos^{-1}\left[\left(\frac{\alpha_1}{P_i\sqrt{c_j}}\right)^{1/2}\right] \quad (15b)$$

As P_i or c_j become large then $A'(P_i, c_j)$ tends to $A(P_i, c_j)$, from eqn (8b).

4.2 3-point bending

With K_I given by eqn (14), $\sigma(x)$ by eqn (12a) and x_m as in eqn (12b), we have:

$$\phi_c = \cos^{-1}\left[\left(\frac{L - x_m}{L - x}\right)^{1/2}\right] \quad (16a)$$

For $P_i \geq \alpha_2/(L\sqrt{c_j})$, the searched area $A'(P_i, c_j)$ is:

$$A'(P_1,c_j) = \int_0^{x_m}\left(\frac{2\phi_c}{\pi}\right)(2b\mathrm{d}x) = \frac{4b}{\pi}\left[L\cos^{-1}\left(\sqrt{\frac{L - x_m}{L}}\right) - \sqrt{(L - x_m)x_m}\right] \quad (16b)$$

We note that both $A'(P_i, c_j)$ and $A(P_i, c_j)$, eqn (12c), tend to $2bL$ as either P_i or c_j become large.

4.3 Hertzian indentation

If the flaws are not oriented perpendicular to the radial direction, then the compressive hoop stress must also be taken into account.

The total tensile stress acting across the crack face, oriented at θ to the radial direction is now:

$$\sigma = \sigma_{rr}\cos^2(\theta) + \sigma_{\theta\theta}\sin^2(\theta) = \frac{(1-2\nu)}{2\pi}\frac{P}{r^2}\cos(2\theta) \quad (17a)$$

With a_{min} as in eqn (13e) and r_m as in eqn (13c) we have:

$$\phi_{c1} = \frac{1}{2}\cos^{-1}\left(\frac{a_{min}}{a}\right) \qquad a_{min} \leq r \leq a \quad (17b)$$

$$\phi_{c2} = \frac{1}{2}\cos^{-1}\left(\frac{r^2}{r_m^2}\right) \qquad a \leq r \leq r_m \quad (17c)$$

The searched area up to a load P_i, $A'(P_i, c_j)$, must be calculated in two parts:

$$A'(P_1,c_j) = \int_{a_{min}}^{a}\left(\frac{2\phi_{c1}}{\pi}\right)(2\pi r\mathrm{d}r) + \int_{a}^{r_m}\left(\frac{2\phi_{c2}}{\pi}\right)(2\pi r\mathrm{d}r) \quad (17d)$$

$$A'(P_1,c_j) = (r_m^4 - a^4)^{1/2} - a_{min}\,a^2 - a^2_{min} \quad (17e)$$

As P_i or c_j become large, the expression for $A'(P_i, c_j)$ does *not* reduce to that for $A(P, c)$ eqn (13f), because the stress state is now different.

If the angular modifications to the searched area, eqns (15b), (16b) and (17e), are not taken into account the searched areas are over-estimated; thus the crack density required to produce a given probability of fracture, $F(P)$, is under-estimated. The calculations below show that this under-estimate can be a large factor, up to $\times 10$.

5 Method of Solution of Eqn (6)

As mentioned above, the solution of eqn (6) is apparently straightforward if one forces $m = n$ so that **A** is a square matrix. Unfortunately, standard routines for inverting matrices show that **A** is rank-deficient unless n is very small, say 3 or 4. A small value for n corresponds to representing $F(P)$ by a few points, which is obviously unsatisfactory. Therefore one chooses m to be large (say 25) and n to be smaller (say 10), in which case the system is over-determined and there is no unique solution; **A** may still be rank-deficient. Singular-value decomposition[16] of **A** may be employed to find a minimal least-squares solution for **x**. Unfortunately, in a number of cases it has been found some of the components of **x**, (i.e. the required flaw densities) are less than zero, which is physically unrealistic. The most satisfactory method has been to use a computer library 'black box' routine, Numerical Algorithms Group E02GBF, which solves over-determined linear equations subject to linear constraints on the unknowns; the routine finds **x** so as to minimise the sum of the absolute values of the residuals **Ax-b**.

6 Computer simulated 3-point bend tests

As a demonstration of the method, we describe the results from a series of computer simulations of three-point bend tests. Semi-circular cracks with randomly chosen depths, positions and orientations were generated. The stress intensity factor for each crack is given by eqn (14). When, under an increasing load, K_I at one crack reached K_{IC}, the load, crack size, position and angle were noted. The whole process was repeated a number of times (typically 100) to give average values. The average flaw spacings for flaws of size c to $c+\Delta c$, $\lambda(c)$, were inputs to the program.

$N(c)$ flaws were distributed in each size range $(c - \Delta c/2) < c < (c + \Delta c/2)$ where Δc was usually 1 μm. The minimum crack size was 1 μm, the maximum 20 μm. The flaws were positioned within a

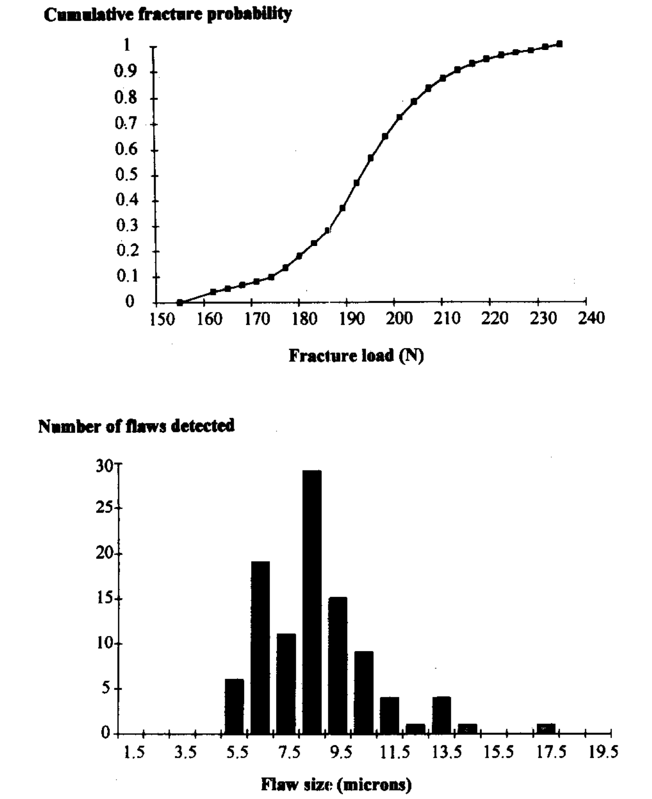

Fig. 5. Results from 100 simulated three-point bend tests: (a) cumulative fracture probability, (b) frequency plot of crack sizes found.

rectangular area, size $2bL$ (Fig. 3), with x-coordinate $-L < x < +L$. Therefore, $N(c) = 2bl/[\lambda(c)]^2$. For $\lambda(c)$ so large that $N(c) < 10$, $N(c)$ was set equal to 10, and cracks generated with x positions $-L' < x < +L'$, where $L' = 10[\lambda(c)]^2/b$.

From the results of 100 tests a cumulative probability of failure curve was constructed, usually at a probability interval of 0·04, i.e. 25 points were chosen. The vector **b** has only 24 components, $-\ln[1-F(P_i)]$; obviously the value $F(P) = 1$ is excluded. The maximum fracture load, P_{max}, sets the lower limit to the crack size, c_{min}, for which it is meaningful to calculate a density: $c_{min} = (\alpha_2/P_{max}L)^2$. The remainder of the cracks, c_j, were spaced 1 μm apart, up to $c_n = 20$ μm. With $m = 24$ values of P_i, and $n = 15$–20 values of c_j **A** and **b** were calculated and NAG E02GBF used to solve for **x**.

Figs 5(a) and 5(b) show the results from 100 simulations; the specimen dimensions were $b = 4$ mm, $h = 1{\cdot}5$ mm and $L = 10$ mm; $Y = 0{\cdot}713$, appropriate to semi-circular surface breaking flaws, and $K_{IC} = 2$ MPam$^{1/2}$. Figure 5(a) shows the cumulative probability distribution, and Fig. 5(b) shows the actual values of the crack sizes detected. (Calculating these crack sizes required knowledge of the crack position.) Figure 6 shows the areas searched for cracks of all sizes during all 100 tests; the upper curve is the searched areas calculated using eqn (12c), the lower curve those areas calculated

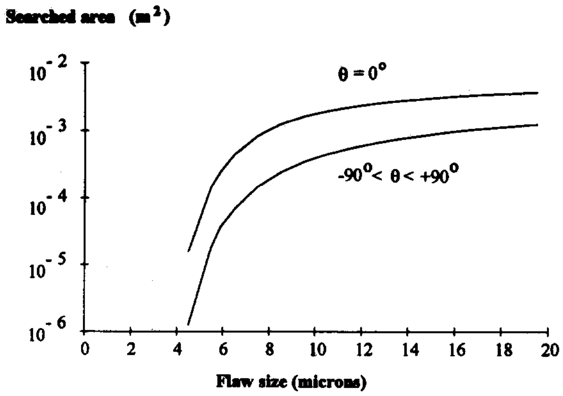

Fig. 6. Areas searched for cracks in 100 tests: upper curve assuming that all cracks are oriented normal to the applied stress; lower curve assuming cracks are randomly oriented with respect to applied stress.

from eqn (16b). The total searched area calculated ignoring the angular variation of K_I with θ is considerably larger than that calculated from eqn (16b), particularly for small crack sizes.

Figures 7(a) and 7(b) show the results of the determination of $n(c)\Delta c$, or equally $\lambda(c)$; clearly $\lambda(c) = 1/\sqrt{n(c)dc}$. Each graph shows three sets of values of $\lambda(c)$: the values input to the computer program, those calculated by Wilshaw's method,[6] and the values returned from the matrix method of calculation. Figure 7(a) shows the results with

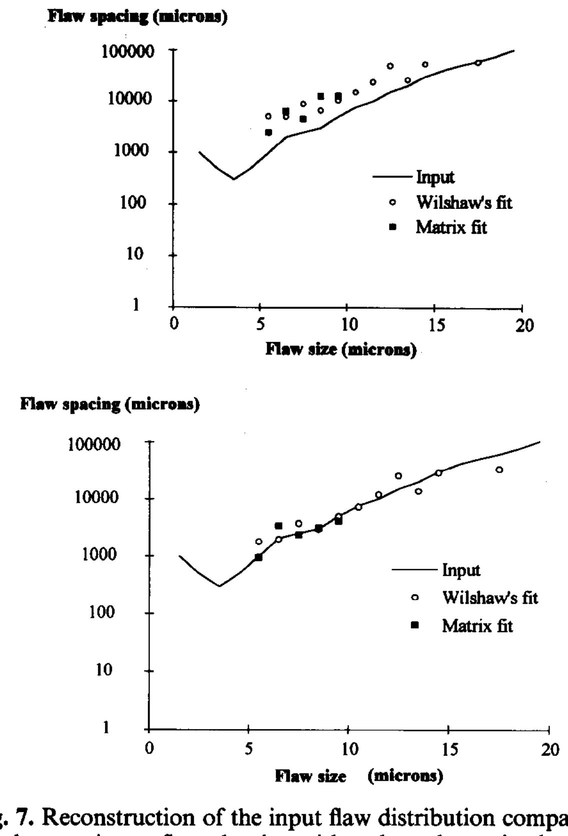

Fig. 7. Reconstruction of the input flaw distribution comparing the known input flaw density with values determined using both Wilshaw's method and the matrix method: (a) Searched areas calculated assuming that cracks are oriented normal to the applied stress. (b) Searched areas calculated assuming cracks are randomly oriented with respect to the applied stress.

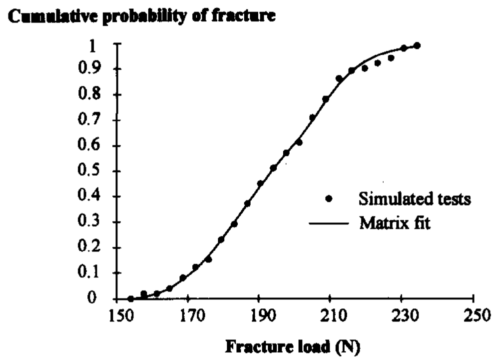

Fig. 8. Reconstruction of cumulative probability of fracture versus fracture load (Fig. 5(a)) using the flaw density distribution calculated by the matrix method.

the searched areas calculated from equation (12c), (ie. all cracks were assumed to lie normal to the tensile stress): the flaw spacings are all too high, i.e. the calculated crack densities are all too small, the discrepancy getting worse as the crack size decreases. Figure 7(b) shows the results with the searched areas calculated from equation (16b). It can be seen that both Wilshaw's method and the matrix method now give accurate estimates of the flaw spacings. Wilshaw's method gives an accurate estimate of the flaw density over almost the entire flaw size range. However, this accuracy is gained at the expense of speed; the position of the flaw actually causing fracture had to be found — to enable the flaw's size to be calculated. In the case of three-pointed bending this may not be a problem, but as mentioned above, this can be difficult in the case of Hertzian indentation. The matrix method of solution gives a reasonably accurate estimate of the flaw density, but only over a limited range of crack sizes. It appears that the best fit for the required components of **b** can be obtained by slightly over-estimating the density of small flaws (around 6–7 microns) and reducing the density of large flaws. This means that this method may not be appropriate for determining flaw densities for 'strength of ceramics' purposes where the large flaws are the ones of interest. The reason for this slight inaccuracy in the estimation of $n(c)\Delta c$ is related to the fact that the searched-area matrix **A** is frequently singular, as mentioned in Section (3(a)). The problem arises for the larger cracks. Figure 6 shows that the searched area no longer varies greatly with the size of the crack for sizes above ~12 μm. This implies that the rows of the matrix **A** may appear to be dependent and the computer routine for inverting the matrix ignores this 'unreliable' data. To compensate, the density of small flaws must be increased somewhat.

Figure 8 shows the results from taking the flaw distribution given by the matrix method, (i.e. the vector **x**) and reconstructing the cumulative probability of fracture curve: $F(P_i) = 1 - \exp(-b_i)$ with $b_i = A_{ij}x_j$. The agreement is quite remarkable.

Conclusions

Using Wilshaw's concept of 'searched area' has enabled a general method for determining the density of surface flaws in brittle materials to be derived, via the solution of a set of over-determined simultaneous equations. Weakest-link statistics are assumed, and it is also assumed that the shape of the cracks causing fracture is known, so that the stress-intensity factor for this crack under any loading situation can be evaluated. The only further inputs necessary are the fracture loads determined from a series of tests. For the three examples presented here (uniform tension, three-point bending and Hertzian indentation, ignoring the stress gradients) analytical expressions for the searched areas can be obtained, including the case where the flaws are assumed to be skewed at an angle with respect to the tensile stress. In general, this will not be possible, and numerical calculations of the searched area will be necessary. The procedure is as follows. First, for a given crack size and a given load it is necessary to calculate the minimum (r_{min}) and maximum (r_{max}) positions within which $K_I > K_{IC}$ at any stage during loading to P, bearing in mind that K_I may be greater than K_{IC} at loads less than P. Secondly, at each of a sufficiently large number of positions within this region, one must calculate the maximum value of K_I for the given crack at any load up to P, and hence determine the maximum angle, ϕ_c, through which the crack can be skewed and still maintain $K_I = K_{IC}$. Then, with the elemental area ΔA:

$$A(P_1, c_j) = \sum_{k=1}^{n} (\Delta A)\left(\frac{2\phi_c(r_k)}{\pi}\right) \qquad (18a)$$

$$r_k = \left(r_{min} + \frac{\Delta r}{2}\right) + (k-1)\Delta r \quad \Delta r = \frac{r_{max} - r_{min}}{n} \qquad (18b)$$

Work is in progress (Warren, Hills and Roberts)[17] applying this method to the determination of the surface flaw densities by Hertzian indentation taking the stress gradients of the Hertzian field into account.

We have deliberately restricted the analysis to consider only surface flaws, because our main interest is in the use of Hertzian indentation for evaluating surface damage following polishing/grinding; in Hertzian indentation the stress field is so strong at the surface that *only* surface flaws are detected. As pointed out by a referee, we recognise that in three-point bend testing, fracture may initiate from volume flaws. In principle, it is a simple matter to repeat the analysis presented here to deal with the case where only volume flaws are present. However, further work is necessary to determine whether meaningful results can be obtained by this matrix method when *both* surface and volume flaws are present.

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Appendix 1

Here we demonstrate that in the case of three-point bending, *not* taking the angular variation into account, the concept of searched areas gives the same results for the flaw density as that originally derived by Matthews *et al.*[8] The integral form of eqn (6) is:

$$\int_{c_1}^{c_2} A(P,c)n(c)\mathrm{d}c = -\ell n[-F(P)] \qquad \text{(A1)}$$

$A(P,c)$ is as given in eqn (12c). The lower limit of the integral, c_1, is the smallest crack that can be detected by a load P: $c_1 = (\alpha_2/PL)^2$. The upper limit of the integral, c_2, is the largest crack that can be detected by a load P, so that by assuming K_I as given in eqn (9) there *is* no upper limit. By differentiating eqn (A1) once with respect to P we obtain:

$$\frac{2b\alpha_2}{P^2}\int_{c_1}^{c_2} \frac{n(c)}{\sqrt{c}}\mathrm{d}c = \frac{F'(P)}{1-F(P)} \qquad \text{(A2)}$$

where ' denotes differentiation with respect to P. By differentiating again we obtain:

$$\frac{4b\alpha_2^2 n(c_1)}{P^4 L} - \frac{4b\alpha_2}{P^3}\int_{c_1}^{c_2} \frac{n(c)}{\sqrt{c}}\mathrm{d}c = \left[\frac{F'(P)}{1-F(P)}\right]^2 + \frac{F''(P)}{1-F(P)} \qquad \text{(A3)}$$

By eliminating the integral term between eqn (A2) and (A3) we obtain:

$$n(c_1)\mathrm{d}c_1 = \frac{P^4 L}{4b\alpha_2^2}\left\{\frac{2F'(P)}{P[1-F(P)]} + \left[\frac{F'(P)}{1-F(P)}\right]^2 + \frac{F''(P)}{1-F(P)}\right\}\mathrm{d}c_1 \qquad \text{(A4)}$$

where $n(c_1)$ is the number of cracks of size c_1 per unit area per unit size range. If we use the notation of Matthews *et al.*, then S_m is defined as the maximum stress corresponding to a load P and $\Phi(S_m)$ is the cumulative probability of failure. By noting that

$$P = \frac{S_m}{\beta}, \quad \beta = \frac{3L}{bh^2} \qquad \text{(A5)}$$

$$F(P) = \Phi(S_m), \frac{\mathrm{d}F(P)}{\mathrm{d}P} = \frac{\mathrm{d}\Phi(S_m)}{\mathrm{d}(S_m/\beta)} = \beta\frac{\mathrm{d}\Phi(S_m)}{\mathrm{d}S_m} \qquad \text{(A6)}$$

$$\frac{\mathrm{d}^2F(P)}{\mathrm{d}P^2} = \beta^2\frac{\mathrm{d}^2\Phi(S_m)}{\mathrm{d}S_m^2} \qquad \text{(A7)}$$

$$c_1 = \left(\frac{\alpha_2 L}{\beta S_m}\right)^2, \qquad \mathrm{d}c_1 = \frac{-2}{S_m^3}\left(\frac{\alpha_2 L}{\beta}\right)^2\mathrm{d}S_m \qquad \text{(A8)}$$

eqn (A4) can be converted to the original result of Matthews *et al.* (their eqn 10), which is written in terms of $g(S_m)$, the number of cracks which give a strength S_m per unit area per unit strength range:

$$g(S_m)dS_m = \frac{S_m}{bL}\left\{\frac{2\Phi'(S_m)}{S_m[1-\Phi(S_m)]} + \left[\frac{\Phi'(S_m)}{1-\Phi(S_m)}\right]^2 + \frac{\Phi''(S_m)}{1-\Phi(S_m)}\right\}dS_m \quad \text{(A9)}$$