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The Accuracy of Failure Predictions Based on Weibull Statistics

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Abstract

The mechanical failure of ceramic materials is usually characterised by probability equations based on Weibull's weakest link theory. To evaluate these for components in a varying stress field, postprocessors have been developed by us as well as others. The effect of the errors which arises from the postprocessing on the fractional errors in the nominal failure load, the probability of failure or the probability of survival are found and discussed.

Introduction

The mechanical failure of ceramics is usually considered to be governed by failure laws which are based on the original ideas of Weibull[1] in 1939. Since then the theory has been developed to allow for multiaxial stress fields and for different failure criteria. Essentially, the ceramic is considered to have many small flaws (cracks) distributed throughout the body and failure depends upon the first crack to fail: hence the concept of the weakest link. These small cracks will have different sizes and will be randomly orientated. The largest crack may not cause failure as it may occur in an area of low stress or it may be orientated so that, for example, the stress normal to the crack is small and hence is less likely to fail when compared to the crack which is orientated so that the stress normal to the crack is high.

To account for these variations, the mechanical failure of ceramics is normally described by a probability distribution and typically the failure probability will be described by the equation

$$P_f = 1 - e\left\{-\left(\frac{\sigma_{nom}}{\sigma_0}\right)^m \sum\nolimits_v\right\} \qquad (1)$$

where Σ_v is a stress volume integral given by

$$\sum\nolimits_v = \int_v \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left(\frac{\sigma_e}{\sigma_{nom}}\right)^m \sin\phi \, d\phi \, d\psi \, dv \qquad (2)$$

for a body with a random distribution of crack orientations.

The stress volume integral accounts for the variation of stress throughout the body and also, through σ_e, the different failure criteria. In these equations, σ_0 and m are material properties which correspond to the strength of the ceramic and to the spread in the failure loads and σ_{nom} is a nominal stress. As well as stress volume integrals in which the stress is evaluated over the volume, there are also failure laws based on the stress distribution over the surface of the component.

In order to determine the failure probability of a structure in which the stress varies, the normal procedure is to perform a finite element analysis and then using the data from this analysis evaluate the failure probability using a postprocessor based on (1) and (2). How our postprocessor worked is described in Ref. 2.

To compare the performance of various postprocessors a numerical round robin was performed and the results have been published by Dortmans *et al.*[3] To compare the errors a measure called $\bar{S}_{nom}$ was used. This is the predicted mean nominal stress and is directly related to σ_{nom} in eqn (1). This was chosen as it is not too sensitive to errors in the postprocessor.

However, a designer is probably more interested in properties such as probability of failure or nominal failure stress for a given probability of failure (or survival). The purpose of this paper is to discuss the relationships between these values and the relationship with the errors arising from the postprocessor when evaluating the stress volume (or area) integral.

Theory

The probability of failure as given by (1) can be generalised so that Σ represents either a stress volume, Σ_V, or stress area, Σ_A, integral, i.e.

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$$P_f = 1 - e\left\{-\left(\frac{\sigma_{nom}}{\sigma_0}\right)^m \Sigma\right\} \quad (3)$$

and for the survival probability

$$P_s = e\left\{-\left(\frac{\sigma_{nom}}{\sigma_0}\right)^m \Sigma\right\} \quad (4)$$

Now, for a particular value of survival probability

$$\sigma_{nom} = \sigma_0\left(\frac{\ln(1/P_s)}{\Sigma}\right)^{1/m}$$

or

$$\sigma_{nom} \propto (1/\Sigma)^{1/m} \quad (5)$$

This gives a relationship between the nominal failure stress, the stress integral and the Weibull parameter. Now, because of errors in the postprocessing

$$\frac{\sigma^*_{nom}}{\sigma_{nom}} = \left(\frac{\Sigma}{\Sigma^*}\right)^{1/m} \quad (6)$$

where the asterix denotes the numerically derived values. This provides a measure of the error in the nominal failure stress which depends on the Weibull modulus and the numerical error from the postprocessing step in calculating the stress integral. Equation (6) also applies to $\bar{S}_{nom}$.

On the other hand, if the survival probability, P_s, is required then from (4)

$$P_s = e\left\{-\left(\frac{\sigma_{nom}}{\sigma_0}\right)^m \Sigma^*\left(\frac{\Sigma}{\Sigma^*}\right)\right\} = P_s^{*(\Sigma/\Sigma^*)}$$

or

$$\frac{P_s^*}{P_s} = P_s\left(\frac{\Sigma^*}{\Sigma} - 1\right) \quad (7)$$

This now provides a measure of the error in the survival probability which depends on the survival probability and the error in the stress integral.

For the failure probability and using (7) gives

$$\frac{P_f^*}{P_f} = \frac{1 - P_s^{\Sigma^*/\Sigma}}{P_f}$$

or

$$\frac{P_f^*}{P_f} = \frac{1 - (1-P_f)^{\Sigma^*/\Sigma}}{P_f} \quad (8)$$

This provides a measure of the error in the failure probability which depends on the failure probability and the error in the stress integral.

Results and Discussion

Equations (6), (7) and (8) have been evaluated to examine the effect of errors in determining the stress integrals and the results are given in Figs 1, 2 and 3.

For the nominal failure stress, Fig. 1 shows the dependency on the Weibull parameter and that the errors in evaluating the stress integral are reduced when calculating the nominal failure stress, particularly for high values of Weibull modulus. However, as has been shown elsewhere,[3] the errors in evaluating Σ are greater for higher values of m.

From Fig. 2 it can be seen that to evaluate the survival probability accurately, it is important for low survival probabilities that the stress volume integral is evaluated accurately. However, for high survival probabilities, which is the realistic region for design, the errors are smaller.

From Fig. 3 it can be seen that a high failure probability reduces the fractional error of failure probability caused by errors in evaluating the stress integral. The error in the failure probability is always lower than that in the stress integral.

The equations given above have been derived using a 2-parameter Weibull model. However, a

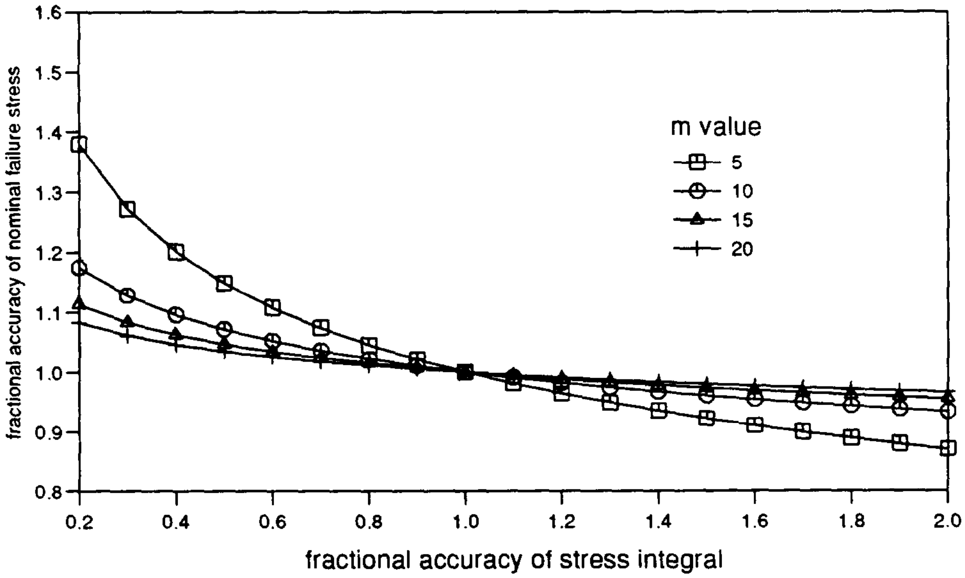

Fig. 1. Accuracy of nominal failure stress.

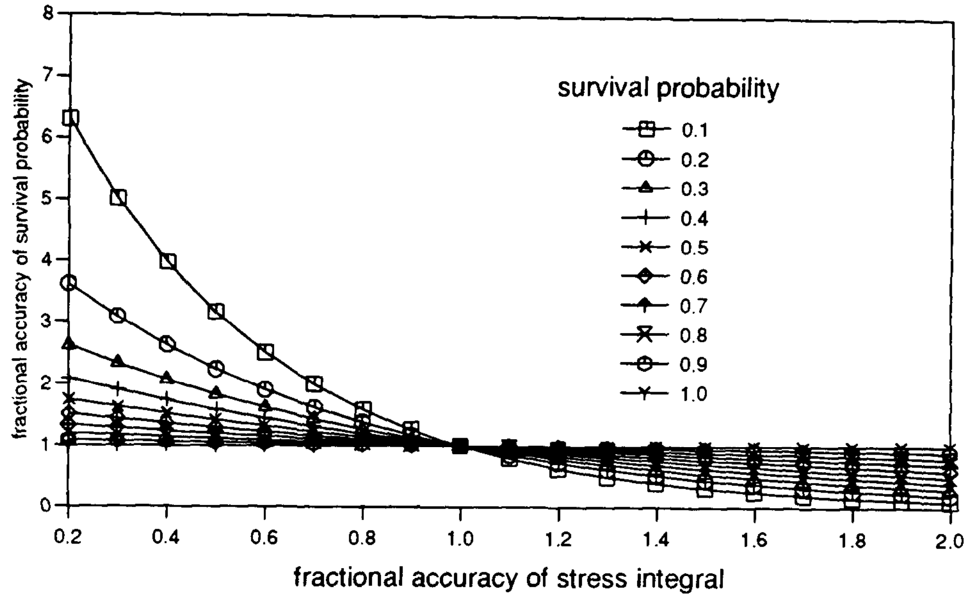

Fig. 2. Accuracy of survival probability.

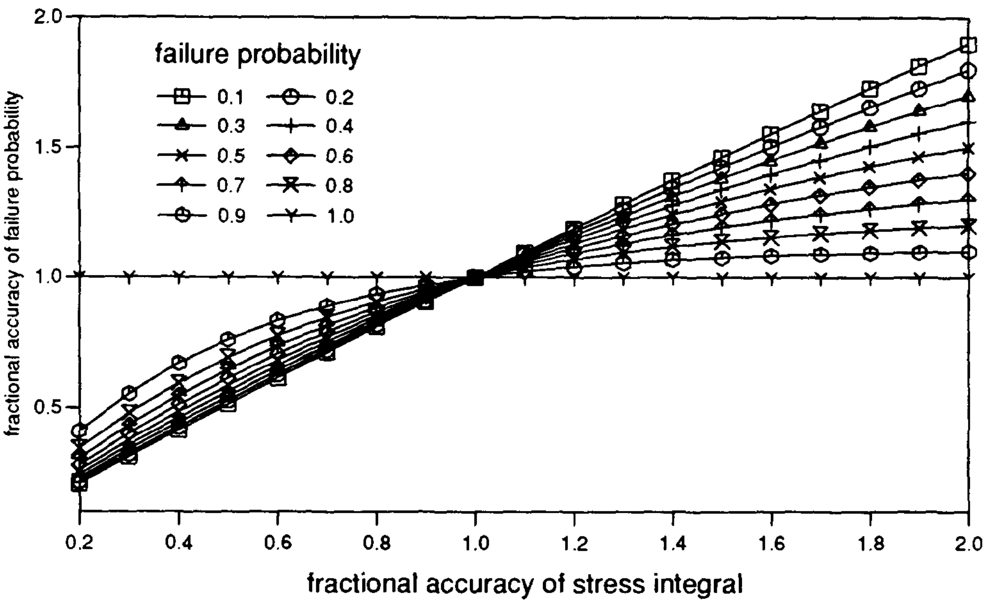

Fig. 3. Accuracy of failure probability.

3-parameter model using a threshold stress is also used in describing the mechanical failure distribution of ceramics. In this case the relationship given by (6) is no longer valid although the relationships (7) and (8) still hold.

A further measure of error is the absolute error, i.e.

$$| P_s^* - P_s | = | P_s^{\Sigma^*/\Sigma} - P_s | \tag{9}$$

Also

$$| P_f^* - P_f | = | (1 - P_s^*) - (1 - P_s) |$$

$$\therefore | P_f^* - P_f | = | P_s^{\Sigma^*/\Sigma} - P_s | \tag{10}$$

It can be seen from (9) and (10) that, not surprisingly, the absolute error in the failure and survival probabilities is the same. This is probably a better indicator of error because, for example, the practical difference between a failure probability of 1×10^{-5} and 2×10^{-5} is negligible even though the fractional error is 50%.

The results of evaluating (9) are shown in Fig. 4 for the region of high survival probability which is the realistic design range.

Equation (10) can be further developed as

$$P_f^* - P_f = (1 - P_f)^{\Sigma^*/\Sigma} - (1 - P_f)$$

For small probabilities of failure and when Σ^*/Σ is approximately 1, then

$$\begin{aligned} P_f^* - P_f &= 1 - \frac{\Sigma^*}{\Sigma} P_f - 1 + P_f \\ \therefore P_f^* - P_f &= P_f \left(1 - \frac{\Sigma^*}{\Sigma}\right) \end{aligned} \tag{11}$$

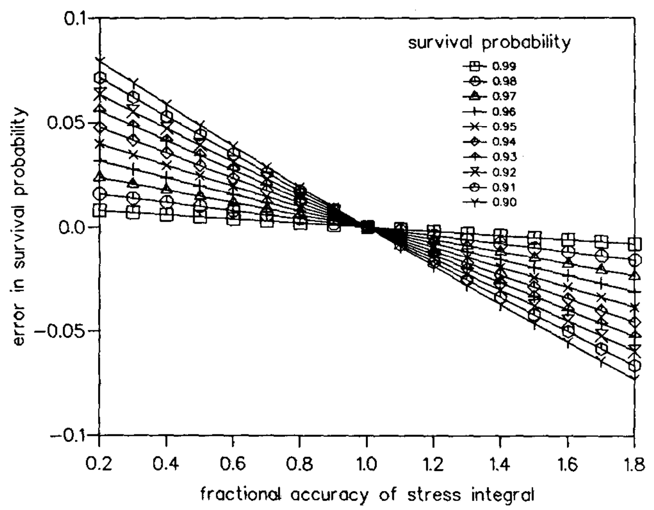

Fig. 4. Error in survival probability.

An examination of Fig 4 shows that this simplified equation is a good representation of the data and will enable the error in the failure probability caused by an error in determining the stress integral to be readily determined.

Conclusions

Relationships have been developed which give the fractional error in the mean nominal failure stress, the survival probability, the failure probability and the error in survival probability caused by errors in evaluating the stress integrals in the Weibull failure laws used for ceramic materials.

These have been evaluated to provide guides for the accuracy required in calculating the stress integrals in postprocessors.

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