Journal of the European Ceramic Society 15 (1995) 909–912

0955-2219/95/$9.50

0955-2219(95)00060-7

The Efficient Integration of the Orientation Integral in Weakest Link Finite Element Postprocessors

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(Received 3 October 1994; revised 2 March 1995; accepted 10 March 1995)

Abstract

In evaluating the failure probability of ceramics it is usually necessary to numerically integrate an orientation integral which accounts for the random orientation of cracks with respect to the stress field. The procedure for performing this efficiently is discussed and recommendations are given.

Introduction

To evaluate the failure probability or nominal failure load of a ceramic component in a varying stress field a postprocessor is normally used. This takes the results from a finite element analysis and evaluates the failure laws which are typically written as[1]

$$P_f = 1 - e\left\{-\left(\frac{\sigma_{nom}}{\sigma_0}\right)^m \Sigma_v\right\} \qquad (1)$$

where Σ_v is a stress volume integral given as

$$\Sigma_v = \int \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left(\frac{\sigma_e}{\sigma_{nom}}\right)^m \sin\phi \, d\phi \, d\psi \, dv \qquad (2)$$

In eqn (2), σ_e and σ_{nom} are the equivalent and nominal stress respectively and the angles ϕ and ψ are defined in Fig 1.

In order to provide some benchmarks and to investigate how various postprocessors work, a numerical round-robin was carried out and a questionnaire circulated by WELFEP.[2] One question asked referred to the numerical integration of (2). Essentially, this equation accounts for the stress variation within the component and for the possibility that a crack may be orientated in any direction with respect to the stress field.

If eqn (2) is examined it can be seen to comprise of 2 parts; an integration over a unit sphere within an integration over the volume of the component. Previously,[1] we have investigated the integration rule within the volume of the component but have not discussed the integration over the unit sphere.

The purpose of this paper therefore, is to discuss the problem of how to efficiently and accurately integrate

$$I = \int_0^{2\pi} \int_0^{\pi} \left(\frac{\sigma_e}{\sigma_{nom}}\right)^m \sin\phi \, d\phi \, d\psi \qquad (3)$$

As this is over a unit sphere and is to account for the orientation of the cracks with respect to the stress field it is often called the orientation integral. In the questionnaire,[2] it was found that 'generally Gauss–Legendre integration is applied with 10, 15 or 20 Gauss points'.

However, the question arises, is this the best method?

Theory

If the maximum principal stress, σ_1, is taken as the nominal stress then (3) can be rewritten as

$$I = \int_0^{2\pi} \int_0^{\pi} f\left(\frac{\sigma_2}{\sigma_1}, \frac{\sigma_3}{\sigma_1}, m, \phi, \psi\right) d\phi \, d\psi \qquad (4)$$

and if the integration is performed numerically then

$$I = \sum_{i=1}^{N} \sum_{j=1}^{N} f_{ij} W_i W_j \qquad (5)$$

where N is the number of sampling points and W_i and W_j are the weighting factors associated with the integration scheme.

For the Gauss–Legendre method, a linear transformation has to be performed to make the limits of integration +1 and −1. The sampling points are non-uniformly distributed and the values of the weighting factors vary.[3]

For the rectangular rule, the weighting factors are constant and the sampling points are evenly distributed through the domain. When integrating in 1-D this is equivalent to dividing the area under the curve into equal strips with the weighting

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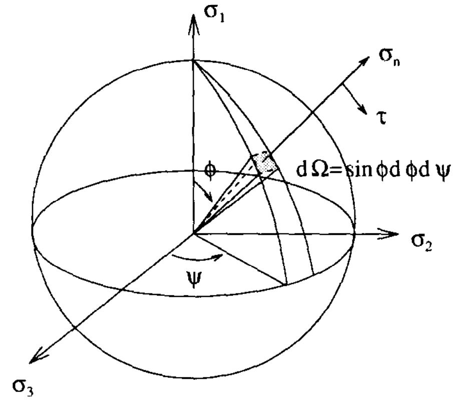

Fig. 1. Unit sphere in principal stress space.

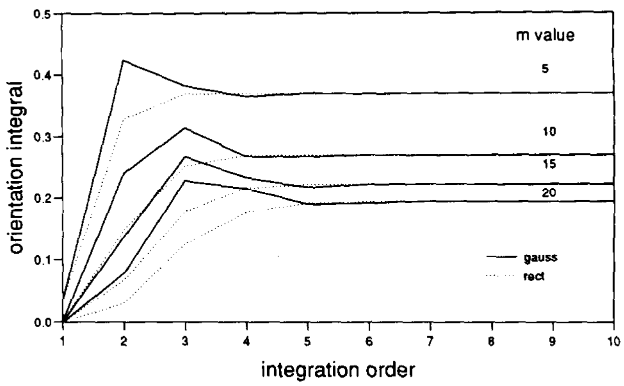

Fig. 3. Convergence of orientation integral for equi-biaxial tension.

factor the strip width and the sampling point being the centre of each strip. (Often, numerical integration is called quadrature and the Gauss–Legendre method is called Gaussian quadrature).

To investigate the efficiency of the evaluation of the integral both the Gaussian and rectangular rules will be used. Also, as (4) shows that the ratios σ_2/σ_1 and σ_3/σ_1 are important, various cases are considered, i.e.:

(i) uniaxial tension $\sigma_2/\sigma_1 = \sigma_3/\sigma_1 = 0$
(ii) equi-biaxial tension $\sigma_2/\sigma_1 = 1; \sigma_3/\sigma_1 = 0$
(iii) tension-compression $\sigma_2/\sigma_1 = 1; \sigma_3/\sigma_1 = -1$
(iv) equi-triaxial tension $\sigma_2/\sigma_1 = \sigma_3/\sigma_1 = 1$

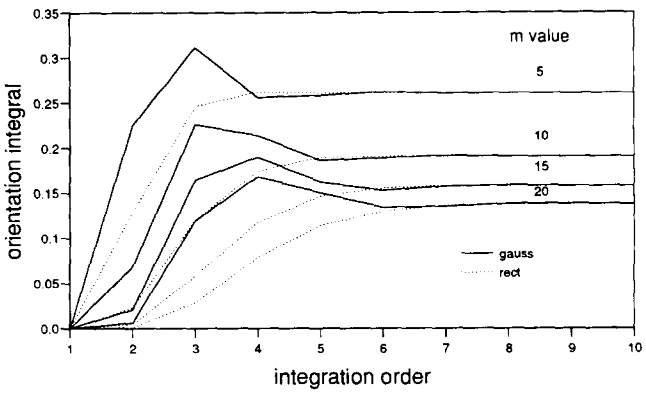

Fig. 4. Convergence of orientation integral for tension–compression.

Results and Discussion

The results are shown in Figs 2–5 for a shear insensitive equivalent stress, i.e. $\sigma_e = \sigma_n$.
In this case

$$\sigma_e = \sigma_1\cos^2\phi + \sigma_2\sin^2\phi\sin^2\psi + \sigma_3\sin^2\phi\cos^2\psi \quad (6)$$

From these figures, it can be seen that the calculated orientation integrals converge with increasing integration order and that, as for the integral over the volume,[1] the higher the Weibull modulus, m, the more sampling points are required for the integral to converge. For both rules, a quadrature order of 10 or less appears to be sufficient for the

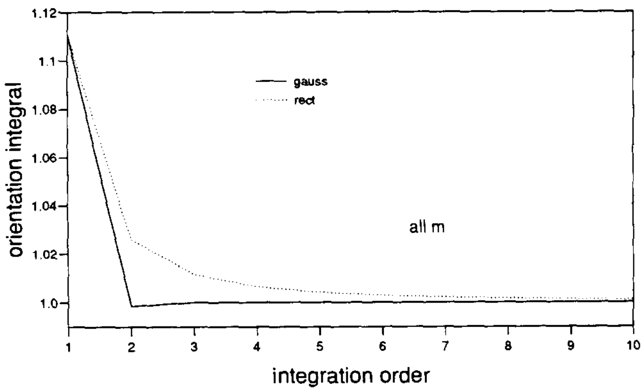

Fig. 5. Convergence of orientation integral for equi-triaxial tension.

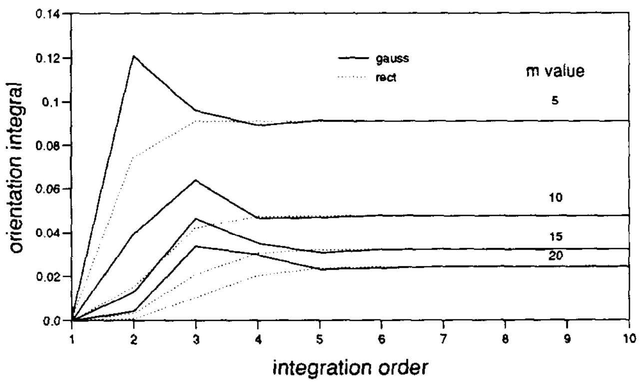

Fig. 2. Convergence of orientation integral for uniaxial tension.

ranges of m values and stress ratios tested. However, in contrast to the calculation of the stress integrals, the rectangular rule can be seen to give a faster convergence than Gaussian quadrature with the exception being the case of equi-triaxial tension.

At first sight, the results may seem surprising considering the superiority of Gaussian quadrature in evaluating the volume integral.[1] However, if the integral, eqns (3) and (6), is examined it can be found to contain terms such as $\cos^m\phi$ and $\sin^m\phi$ because of the stress transformations. When plotted, these functions have shapes similar to a ramp function as shown in Figs 6 and 7. If the

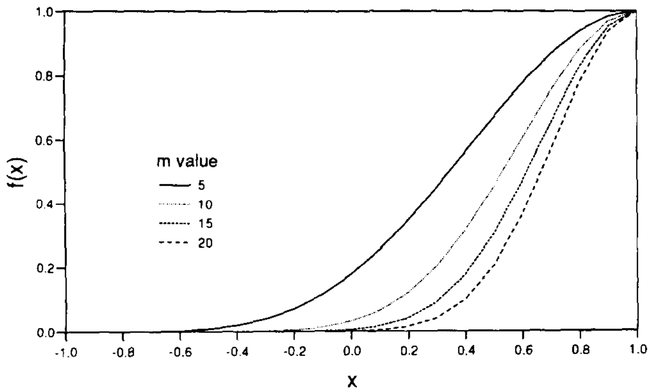

Fig. 6. $f(x) = \sin^m \frac{\pi}{4}(x + 1)$

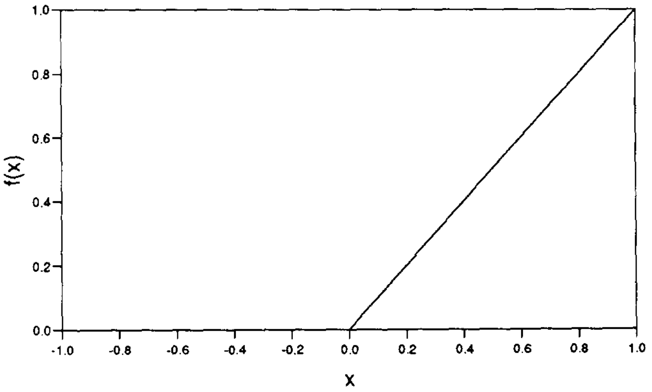

Fig. 7. Ramp function.

rectangular rule is used to integrate the ramp function, the use of 2 or any other even number of sampling points will give the correct answer. On the other hand, when the number of sampling points is odd a higher integration order is required to obtain a converged result. Figs 8 and 9 compare the performance of the rectangular and Gaussian rule in integrating the functions in Figs 6 and 7 and it can be seen that in both cases the rectangular rule is more efficient.

For a shear sensitive criterion, e.g.

$$\sigma_e = (\sigma_n^4 + 6\sigma_n^2\,\tau^2 + \tau^4)^{1/4} \tag{7}$$

similar results are obtained, although the convergence is not so rapid with the value of the orienta-

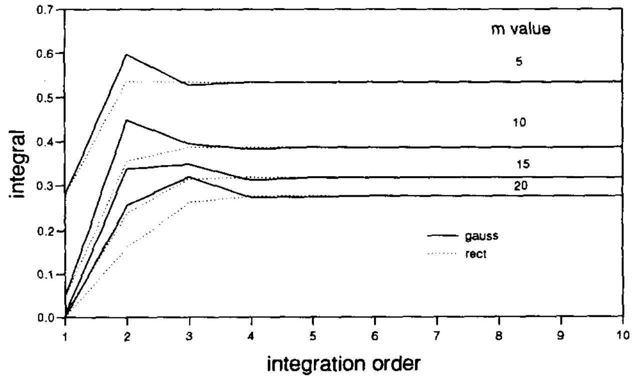

Fig. 8. Integration of $\sin^m \frac{\pi}{4}(x + 1)$

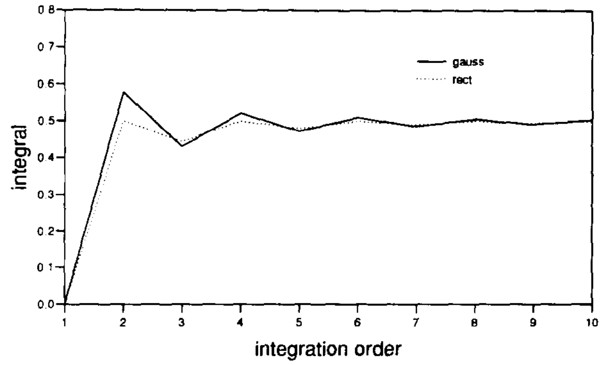

Fig. 9. Integration of ramp function.

tion integral oscillating more before converging. This is particularly so for case (iii), i.e. tension–compression and for $m = 5$. Convergence is also affected by the way in which negative values of stress are treated. If

$$\sigma_e = 0 \text{ for } \sigma_n < 0$$

then convergence is not so good and higher integration orders may be needed.

The effect of any errors in calculating the error in probability of failure, probability of survival or nominal stress has also been examined by the authors.[4] For high survival probabilities, the fractional error in the survival probability is considerably less than the fractional error in the stress integral. It was shown that

$$\frac{P_s^*}{P_s} = P_s^{\left(\frac{\Sigma^*}{\Sigma} - 1\right)}$$

where P_s is the survival probability, Σ is the value of the stress integral and the *superscript denotes the numerically derived values. So, for example, for a fractional error of 10% in Σ, i.e. $\Sigma^*/\Sigma = 1{\cdot}10$, then P_s^*/P_s for a survival probability of 0·99 is 0·999, i.e. a 0·1% error. Alternatively, the absolute error is given as

$$|\,P_s^* - P_s\,| = |\,P_s^{\Sigma^*/\Sigma} - P_s\,|$$

and the absolute error is 0·000994.

In view of these results, an integration order of 10 is recommended as sufficient for practical applications and there is little difference between Gaussian and rectangular quadrature.

Conclusions

To evaluate the orientation integral in the failure laws for ceramic materials, it is recommended that an integration order of 10 is used. The rectangular rule and Gaussian quadrature give almost identical results.

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