Journal of the European Ceramic Society 16 (1996) 1209–1212

0955-2219/96/$15.00
PII:S0955-2219(96)00047-7

The Median Crack Driven by a Point Force

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(Received 13 December 1995; revised version received 16 February 1996; accepted 20 February 1996)

Abstract

The shape of a median crack, driven by a point force, is deduced. It is shown that if it is assumed that self-arrest occurs when the crack tip stress intensity factor falls below the fracture toughness, a near-circular crack may be expected to develop. A universal calibration for the crack tip stress intensity factor is given. Copyright © 1996 Elsevier Science Ltd

Introduction

When a sharp indenter is pressed into a brittle material, median cracks are observed to develop on planes normal to the free surface and aligned with the vertices. This test provides the possibility of determining the fracture toughness of a brittle material and other useful surface information. It is experimentally observed that median cracks form during the loading phase of indentation, but often extend during the unloading phase,[1,2] clearly propelled by residual stresses left by plasticity arising at the tip of the indenter. Recently, both Giannakopoulos *et al.*[3] and Murakami and Matsuda[4] have produced very thorough solutions to the problem of indentation of an elastic–plastic solid by a Vickers pyramid. Each used the finite element method, the former incorporating special elements to allow for the singular stress state arising along the vertices, to produce an accurate, versatile solution. Neither, however, solved the unloading phase, so that the residual stress state remaining upon removal of the indenter has not been found. This is one of the two obstacles remaining before a physical interpretation of the results of the test can be used quantitatively. The second is that a precise calibration for the crack tip stress intensity factor arising around the crack front is needed. In this paper we intend to address the second problem, and, in the absence of a solution for the residual stress state, we shall use an idealized contact pressure distribution. We shall therefore revert to an idealization of the contact load as a point force, as employed in an earlier analysis[5] and by Murakami and Sakae.[6]

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Crack Shape

We will idealize the material being indented as a linear elastic material, and the contact pressure, in reality distributed over a small finite area, approximately square in shape, by a point force. By St. Venant's principle we expect this idealization to be satisfactory beneath the surface, but to be rather poor in the immediate neighbourhood of the contact patch. A corollary of using a point force to represent the contact pressure is that the effects of surface friction may not be included, and we will also assume that only a single crack grows in the median plane. A further consequence of using the point force as the source of contact loading is that the surface layers are put into a state of compression, i.e. $\sigma_{\theta\theta} < 0$ above a conical region, making an angle of approximately 38° with the surface, as depicted in Fig. 1. This highlights the influence of residual stresses in propelling cracks in real materials showing some plastic behaviour.

In our earlier analysis of the problem[5] we noted the form of contours of the stress component $\sigma_{\theta\theta}$ induced by a point force, using Boussinesq's solution,[7] where the coordinate set of Fig. 1 is employed. From this, we idealized the median crack as a circular arc with two tangential straight lines, as shown in Fig. 1. Crack tip stress intensity factors arising around the crack front were then found using the eigenstrain procedure. The problem was subsequently reanalysed by Murakami and Sakae[6] who used an alternative idealized crack shape in the form of a semicircle, but permitted crack closure adjacent to the surface. Thus, one boundary of the open part of the crack was prescribed, but the other, the line of closure, became an unknown of the problem.

In this paper, we wish to adopt an alternative algorithm for determining the shape of the crack:

as the contact load is increased the stresses in the vicinity of the contact patch increase until at a particular flaw the critical stress intensity factor will be reached and the crack will extend. As it does so, the increase in size of the crack will *increase* the crack tip stress intensity, but at the same time the crack front is growing into a region of *decreasing* stress. Eventually, the stress intensity factor falls below the fracture toughness value again. At this point, because of the basic principle of reversibility of the crack extension criterion in linear elastic fracture mechanics,[8] self-arrest will occur. Thus, the final shape of the crack is one where the crack tip stress intensity factor around the crack front is constant, and equal to the material's fracture toughness. This argument assumes that quasi-static extension of the crack occurs, and that crack growth occurs under elastic conditions.

Formulation

The eigenstrain method was used in the analysis. A full introduction to this procedure is given in Ref. 9, but a brief résumé of the principles will be included here. It is a three-dimensional technique, well suited to the analysis of cracks present in steep stress gradients, but a drawback is that only relatively simple remote boundaries may usually be included, such as an infinite space, half-space or a layer. In the present problem we will employ the half-space, and first find the solution for a point force, exerted at an arbitrary point within the solid. From this, the solution for a point *displacement discontinuity* may be found, and this, in turn, is used to build up the stress field associated with an eigenstrain element. The eigenstrain element may be viewed in different ways, but one physical interpretation is to think of its boundary as forming a Somigliana dislocation loop, where the Burgers vector may vary from point to point around the edge. Indeed the idea of a 'shape function', from finite element theory, is used to prescribe the form of the variation of the strength of the dislocation throughout the element, which may normally be either piecewise constant or piecewise linear. An influence function is defined, which relates the strength of the Burgers vector at any general node to the magnitude of the stress state induced at a general point. Hence, a matrix is established giving the resultant stress within each eigenstrain element in terms of the strength of all the elements used to model the crack. That stress resultant is set equal and opposite to the tractions induced by the far field loading *in the crack's absence*, so that the crack faces are thereby rendered stress-free. The following additional points about the method may be noted.

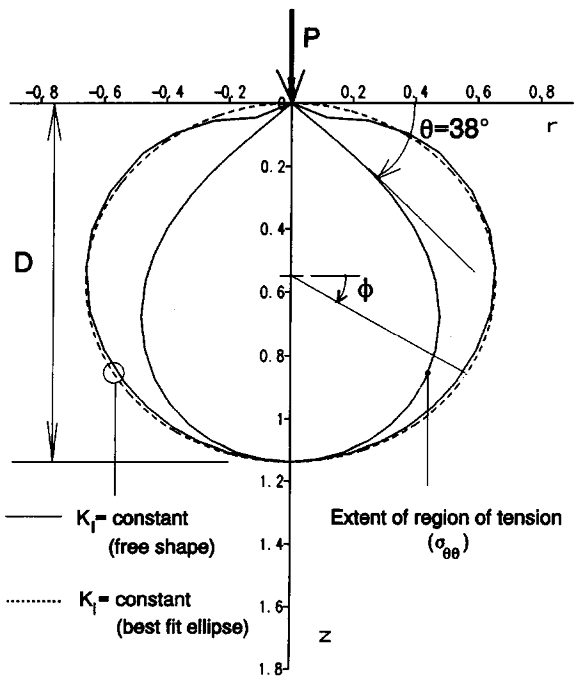

Fig. 1. Geometry of the problem, region where $\sigma_{\theta\theta}$ is positive, and possible crack front shapes.

(1) As both the formulation for an individual element and the nominal stress state in the crack's absence are both solved in closed form for the body as a whole, all of the remote boundary conditions (such as the free surface of the half-space remaining traction-free) are automatically satisfied.
(2) In addition to the basic shape function described, crack front elements have a further form of variation of the internal Burgers vector variation imposed on them; this is to ensure that the displacement and stress field follow the required asymptotic forms along the crack front, which accelerates convergence of the solution.
(3) Because the far-field conditions are satisfied exactly, and only the crack itself needs to be modelled, the solution converges very much more quickly than a corresponding classical finite element solution, and requires far fewer degrees of freedom, for the attainment of a given level of accuracy.

These attributes mean that the technique is ideal for the problem we are addressing here, where the overall geometry is quite simple, but the stresses arising on the plane of the crack, in its absence, vary rapidly with position. Piecewise linear triangular elements were employed,[10] and the asymptotic behaviour required at the crack front was incorporated into adjacent elements only. A uniform (as opposed to graded) mesh was employed, as it has been found that this produces the best convergence.[4] An example of a mesh for the $\sigma_{\theta\theta}$ = constant case is shown in Fig. 2.

Two forms of the possible crack shape were analysed. First, the assumed shape of the crack was that of an ellipse, with the ellipticity left as a free variable, which was adjusted so as to minimize the variation of K_I from point to point around the free edge. Surprisingly, this gives a very good approximation to the constant K requirement, and the variation is very much less than when the crack front was made to follow the contour delineating a constant tension.

To improve the shape of the crack further, the geometric restriction that it be elliptical was removed, and free meshing was employed. The crack was then 'grown', by incrementing the crack size at each point around the crack front, in proportion to the local stress intensity factor found, i.e.

$$\delta c = AK_I^n$$

where A, n are arbitrary constants, the former dimensional, chosen so that the actual increase in crack size is only a small percentage of the current value. This process was carried out several times, until it became clear that the crack was developing in a self-similar manner, i.e. without change in shape, and the stress intensity factor found was uniform around the edge, at all points save the point of application of the load itself, which is singular. It should be noted that, as the contact loading is idealized by a point force, there is no intrinsic length scale in the problem, so that when once a steady state has been reached, the shape of the crack may be expected to remain geometrically similar, i.e. it will develop in a self-similar way. This is, of course, a consequence of the idealized contact loading; had a finite contact patch been employed this would not be true. Another feature of the geometry of the model is that no mode II or III crack tip stress intensity factor can arise, because of the symmetry of the configuration.

Results

The optimal ellipticity found for the constrained crack shape was D/a = 0·85, where D is the depth of the crack and a is the breadth parallel with the surface, as shown in Fig. 1 by a broken curve. The variation of normalized stress intensity factor occurring around the front is shown in Fig. 3. It may be seen that, excluding a very small zone immediately beneath the point of application of the load, where the state of stress is singular, the stress intensity factor varies by up to 25%.

After permitting incremental growth of the model crack, the shape also shown in Fig. 1 was found. The variation of stress intensity factor around the front is shown in Fig. 3; if we again avoid the point of loading, the variation in crack tip stress intensity factor is now less than 8%. Also included in the figure, for completeness, is the original variation of crack tip stress intensity around the crack front associated with a crack shape delineating a constant value of $\sigma_{\theta\theta}$, such as the $\sigma_{\theta\theta}$ = 0 curve shown in Fig. 1. It should be noted that there is no intrinsic length dimension in the problem, save that of the depth of crack, D, itself. Thus the normalized stress intensity factor found, $K_I/P\sqrt{\pi D} \approx$ 0·04, is a universal value, and this means that a knowledge of the final depth of

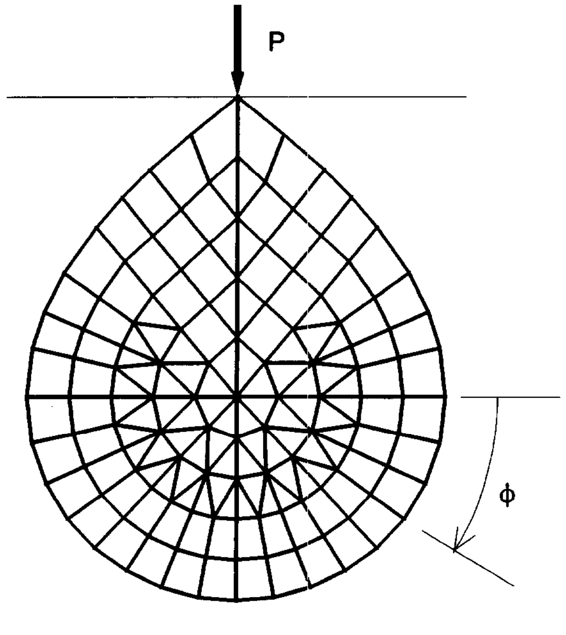

Fig. 2. Example mesh used in one part of the eigenstrain analysis.

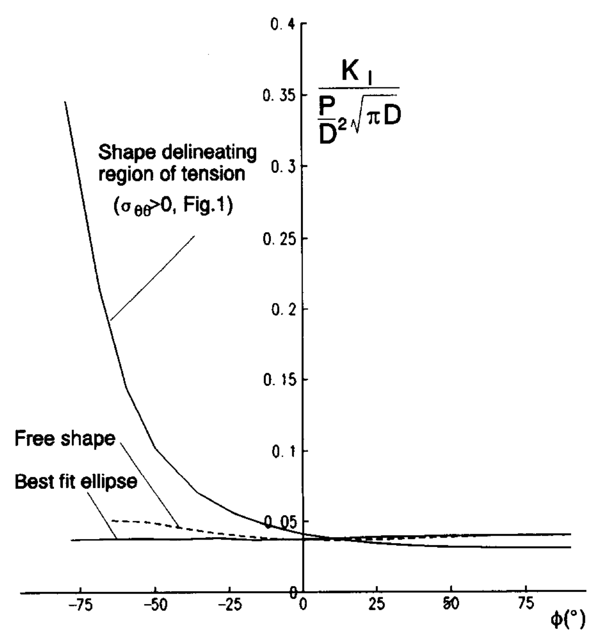

Fig. 3. Variation of crack tip stress intensity factor around crack front for the three possible crack shapes shown in Fig. 1.

the crack, together with the applied load present, is sufficient to determine the fracture toughness, with the assumptions stated at the outset. In practice, as has been stated, removal of the load may well give rise to further crack growth, propelled by residual stresses, but it is felt that the result just presented has merit in its own right, as it provides a rigorous calibration to the idealized problem posed. If an indentation test is stopped without removing the load, and the crack depth measured, the calibration is accurate, save for the area in the immediate vicinity of the contact, where the finite contact pressure distribution and plastic zone have been ignored.

The next step in providing a rigorous solution to the problem posed will be a re-evaluation of the crack shape for the case when the contact pressure distribution is modelled precisely, and the material is given an elastic–plastic stress–strain law. We feel that the bulk of the crack extension will occur upon removal of the indenter, so that propulsion is provided by the residual stress field. For 'nearly brittle' material such as ceramics and glasses these residual stresses will be quite high, as there will be a considerable re-distribution of the load compared with the elastic solution, but at the same time crack-front plasticity will be quite limited, so that the eigenstrain procedure may again be used to advantage in inferring both the crack front shape and the resultant stress intensity factor.

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