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Contact Fracture of Brittle Materials: A Comparison of 2-D and 3-D Fracture Mechanics Solutions

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Abstract

The results of calculations of the mode I stress intensity factors associated with surface-breaking semi-elliptical cracks situated close to Hertzian contacts are presented. The ellipticity of the crack is shown to have a pronounced effect on the magnitudes of calculated surface and crack-bottom K_I *values; the implications of this for determining the fracture toughness of brittle materials via Hertzian indentation are discussed, as is the stability of the ensuing fracture.*

1 Nomenclature

a	Hertzian contact radius
$b_z(x',z')$	Burgers vector density for 3-D calculations
$B_r(z)$	Burgers vector density for 2-D calculations
c	crack depth
C	dimensionless constant calculated from maximum value of stress intensity factor
d	crack length at surface of substrate
δa	incremental change in Hertzian contact radius
δP	incremental change in applied load
E_1, E_2	Young's modulus for sphere and substrate respectively
E^*	combination of elastic constants of sphere and substrate
G	shear modulus
$H(x,z,x',z')$	bounded function for 3-D calculations
K_I	mode I stress intensity factor
K_{IC}	fracture toughness
$K(z,r)$	kernel of integral equation for 2-D calculations
κ	Kolosov's constant
μ	stress intensity factor normalised with respect to crack depth and peak Hertzian pressure
ν_1, ν_2	Poisson's ratio for sphere and substrate, respectively
P	load applied to sphere
P_{fmin}	threshold load for Hertzian fracture
p_o	peak Hertzian pressure
R	sphere radius
S	domain of integration for 3-D calculations
$\sigma_{ij}(z,r), \sigma_{yy}(x,z)$	radial Hertzian stress fields for 2-D and 3-D calculations, respectively

2 Introduction

Localised fracture of brittle materials frequently accompanies surface indentations, where either a blunt or a sharp object presses into a surface. The simplest situation of this type to analyse is the so-called Hertzian[1] contact where a relatively large sphere (radius greater than ~1 mm) is pushed into a surface, close to a pre-existing crack. If there is no localised plasticity then the stress field developed was first determined by Huber;[2] fracture mechanics analysis can be used to calculate the associated stress intensity factors and fracture criteria. This is a challenging problem because of the very steep stress gradients that exist close to surface contacts and also because of the presence of the free surface itself. Numerous solutions to this problem exist in the literature: see for example the work of Frank and Lawn,[3] Wilshaw,[4] Warren,[5] Mouginot and Maugis[6] and Warren *et al.*[7] Most of this previous work has made the simplifying assumption that the cracks are plane—i.e. the fracture mechanics problem is a 2-dimensional one. Here, we use the eigenstrain technique[8] to calculate the stress intensity factors for semi-elliptical and semi-circular surface-breaking cracks close to a Hertzian contact and compare the results with the simpler 2-D formulation.

The primary motivation for this study is the use of Hertzian indentation to evaluate the fracture toughness, K_{IC}, of brittle materials.[9] Because of the steep gradients in the radial component of stress that exist close to a Hertzian contact the stress intensity factor, K_I, for cracks is not a monotonically increasing function of crack length nor of distance from the centre of the contact (Fig. 1). Short cracks are associated with a small value of K_I simply because they are short, while long cracks are also associated with small values of K_I because the crack-tip will now lie in a region of greatly reduced radial stress or even *compressive* radial stress. This implies that there is a *maximum* value of K_I for a crack of intermediate length; this maximum in K_I corresponds to a *minimum* load for crack propagation. Hence, by performing a number of Hertzian fracture tests on a well-abraded ceramic surface (so that there is a large density of pre-existing flaws) and determining the threshold load below which no fracture is ever observed, K_{IC} can be determined. If the radius of the sphere used in the tests is R and ν_1, ν_2 and E_1, E_2 are Poisson's ratio and Young's modulus for the sphere and ceramic, respectively, then the minimum load for fracture is[9] P_{fmin}:

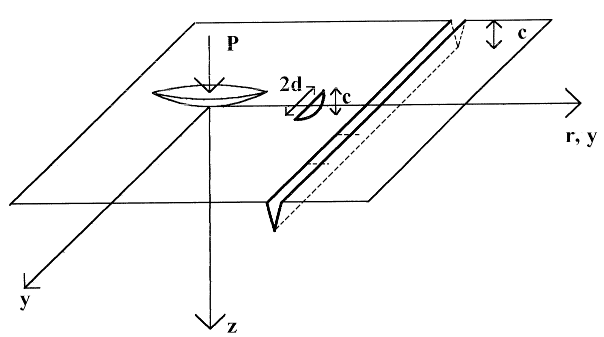

Fig. 1. Hertzian contact: geometry and coordinate systems for the 2-D and 3-D calculations. Under a load P a sphere makes contact on a substrate. Close to the contact patch is a crack. For the 2-D case the crack is an infinitely long straight crack of depth c, we use radial coordinate r and z is measured into the substrate. For the 3-D case the crack is semi-elliptical, with a surface trace of length $2d$ and a depth c. We define the ellipticity as d/c, the co-ordinate system is rectangular

$$\frac{K_{IC} = \sqrt{E^* P_{\text{fmin}}}}{\text{CR}} \tag{1}$$

Here

$$1/\text{E}* = \left(1 - \nu_1^2\right)/\text{E}_1 + \left(1 - \nu_2^2\right)/\text{E}_2 \tag{2}$$

and C is a dimensionless constant whose value depends on Poisson's ratio of the substrate, for the case where the sphere and substrate are made of the same material. For example, for glass with $\nu = 0{\cdot}25$ then for a plane crack $C = 3131$. Experimentally,[9] using a glass sphere of radius $R = 2{\cdot}5$ mm on an glass substrate lapped with 600-grit SiC the minimum load for fracture was 105 N, giving a calculated fracture toughness $K_{IC} = 0{\cdot}8$ MPam$^{1/2}$ a value that compares well with results obtained by more conventional methods: typically a value of 0·7 MPam$^{1/2}$ is quoted. The advantage of using this Hertzian method is that no measurement of any crack size is necessary—because the method relies on performing enough tests to be sure of finding one crack (situated at the right position) for which K_I is a maximum. The value of the constant C also depends on the *shape* of the crack—because clearly the maximum value of K_I is crack shape dependent. Therefore, we need information about the crack shape dependence of C to be able to use eqn (1) for calculating K_{IC} with confidence.

We now proceed to describe briefly the method of calculation of stress intensity factors for surface breaking cracks in steep stress gradients. First, we describe the method appropriate for planar, 2-D, cracks, that of the distributed dislocation technique.[10] In this method, a short plane crack of depth c, normal to the free surface is placed close to the contact zone. The state of stress in the crack's absence is found. When the crack is inserted, unsatisfied tractions appear along the line of the crack. These may be cancelled by the application of equal and opposite tractions along the crack faces which may be generated by installing a distribution of dislocations. These dislocations are not real dislocations in a crystalline lattice, but a mathematical device. The state of stress induced by one of these dislocations in a half-plane is known—the expressions are given explicitly in Ref. 10. By applying a distribution of dislocations, of unknown density $B_r(z)$ the requirement that the crack faces be traction free may be ensured by writing

$$0 = \tilde{\sigma}_{ij}(z, r) + \frac{G}{\pi(\kappa + 1)} \int_0^c B_r(z) K(z, r) \mathrm{d}z \tag{3}$$

G is the shear modulus of the material, κ is Kolosov's constant ($= 3 - 4\nu$ in plane strain), the co-ordinates r, z are as in Fig. 1; the function $K(z, r)$ is given in Ref. 10. The unknown quantity, $B_r(z)$, is defined by $B_r(z) = \mathrm{d}b_r/\mathrm{d}z$, and $\tilde{\sigma}_{ij}(z, r)$ represents the state of stress induced by the contact in the crack's absence—i.e. in this case the radial component of the Hertzian stress field. By using a standard numerical quadrature this integral equation is reduced to a set of linear algebraic equations, typically 20; the mode I stress intensity factor can then be expressed in terms of the unknown coefficients in these linear equations. A standard computer library routine for the solution of simultaneous equations is then employed.

If the radial Hertzian stress ($= \tilde{\sigma}_{ij}(z, r)$ is expressed in terms of the peak Hertzian pressure, p_o, then the result of the calculation is a number μ which is related to K_I by:

$$\mu = \frac{K_I}{p_o\sqrt{\pi c}} \text{ where } \mu = f\left(\frac{r}{a}, \frac{c}{a}, \nu\right) \tag{4a}$$

where a is the radius of the Hertzian contact, $a = (3RP/4E^*)^{1/3}$ and $p_o = 3P/2\pi a^2$ and P is the applied load.

Normalizing with respect to a rather than c we have

$$\frac{K_I}{p_o\sqrt{\pi a}} = \mu\sqrt{c/a} \tag{4b}$$

For the case of semi-elliptical or semi-circular cracks the principle of the calculation is identical. First, place a crack close to the contact and calculate the stress in the crack's absence, then cancel the unsatisfied tractions that arise by the introduction of a distribution of strain nuclei. The strain nuclei chosen are infinitesimal dislocation loops and the resulting integral equation is

$$0 = \tilde{\sigma}_{yy}(x, y) + \frac{G}{4\pi(1-\nu)}\iint_S \left[\frac{1}{r^3} + H(x, z, x', z')\right] \times b_z(x', z')\mathrm{d}S \tag{5}$$

where S is the domain occupied by the crack, and we use rectangular coordinates as shown in Fig. 1. $H(x, z, x', z')$ is a bounded function that accounts for the presence of the free surface. This equation is a hypersingular integral equation, singular to the third degree, in the unknown function $b_z(x', z')$ which represents the crack opening displacement. It is not possible to solve this equation analytically so the domain S is split up into a number of smaller elements and the Burgers loop strength is required to vary from point-to-point according to some prescribed shape function. The simplest one which may be chosen is a piecewise constant distribution. In this case the integral equation is replaced by a family of simultaneous equations which may readily be solved and the (normalised) stress intensity factor found as before. Further details are given by Hills *et al.*[8]

3 Results

Figure 2 shows the results from the 2-D calculation for a material with a Poisson's ratio 0·25. We plot the normalised stress intensity factor versus normalised crack position for various values of normalised crack size. From the graph we see that there is a maximum in the stress intensity factor for a crack of size $c/a \sim 0{\cdot}04$ at a position $r/a \sim 1{\cdot}20$, more accurate values are $c/a = 0{\cdot}044$, $r/a = 1{\cdot}189$. For this crack $K_I/p_0\sqrt{\pi a} \sim 0{\cdot}01829$. The value of the constant C is given by[10] $C = \pi/3/0{\cdot}01829^2 = 3131$.

Figure 3 shows the results from the 3-D calculation for a material with Poisson's ratio 0·25. Again, we plot the normalised stress intensity factor versus normalised crack position for various values of the normalised crack size. The values of stress intensity factor are those *at the base of the crack* and are for cracks of varying ellipticity as indicated in the caption. In this case, for each ellipticity we again have a maximum in the stress intensity factor. In each case the maximum occurs at $c/a \sim 0{\cdot}04$ at r/a 1·2 as in the 2-D case but we note that the value of the maxima are considerably

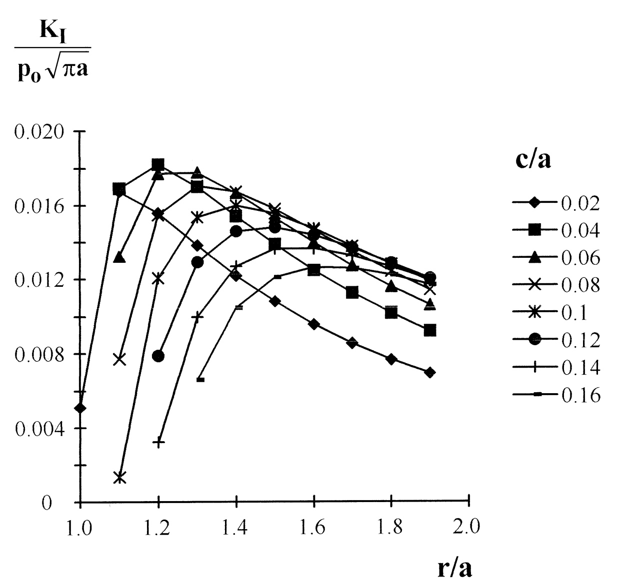

Fig. 2. Results of the 2-D calculation. Normalised stress intensity factor versus normalised crack position for various normalised crack sizes. Poisson's ratio= 0·25.

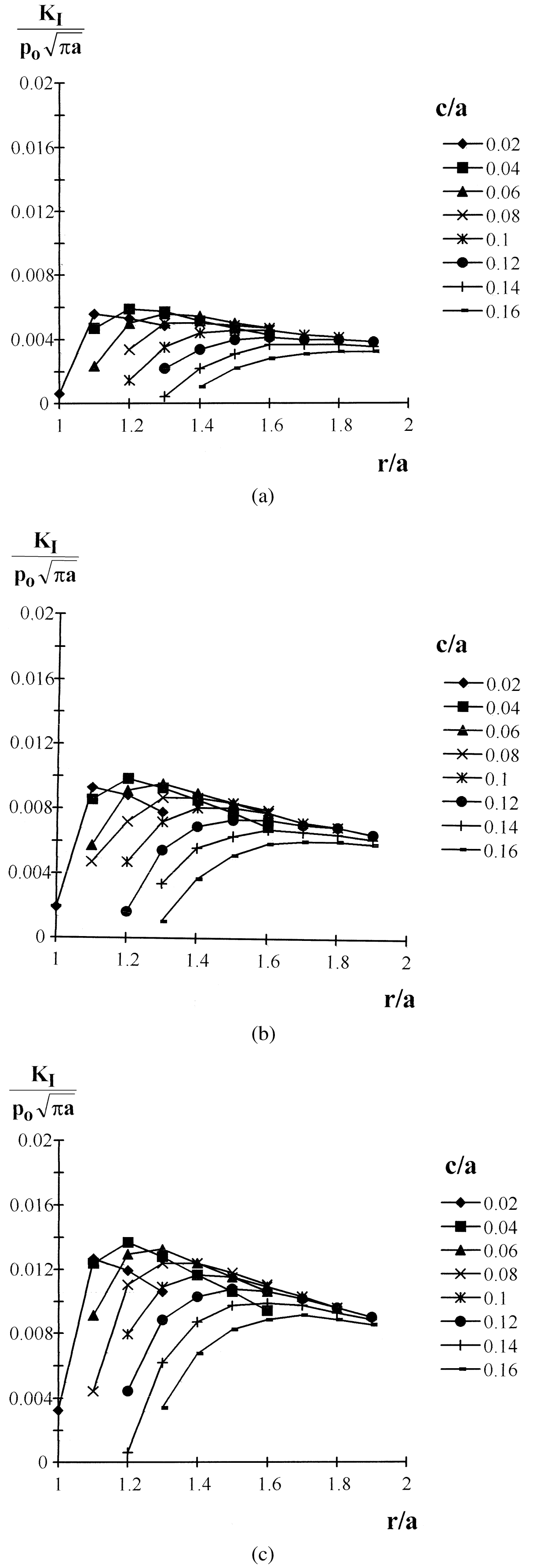

Fig. 3. Results of the 3-D calculation. Normalised stress intensity factors at the base of the crack versus normalised crack position for various normalised crack sizes. Poisson's ratio $= 0{\cdot}25$. (a) Ellipticity $= 0{\cdot}5$, (b) ellipticity $= 1{\cdot}0$, (c) ellipticity $= 2{\cdot}0$.

smaller for the 3-D solution: for $d/c = 0{\cdot}5$ the maximum is $\sim 0{\cdot}006$, for $d/c = 1{\cdot}0$ it is $\sim 0{\cdot}010$ while for $d/c = 2{\cdot}0$ it is $\sim 0{\cdot}014$. This in turn implies values for the constant C of ~29089, ~10472 and ~5343, respectively, cf. C = 3131 for the case $d/c \rightarrow \infty$. It can thus be seen that the value of C is extremely sensitive to the assumed crack shape.

Figure 4 shows similar results from the 3-D calculation except that the stress intensity factors are now evaluated at points *close to where the crack breaks the surface*. To evaluate the stress intensity factors actually *at* the surface is impossible in principle because the degree of singularity here no longer shows the characteristic square-root behaviour.[8] As described in the Introduction, in the 3-D calculation the crack is split up into a number of smaller elements, typically 40. The results presented here are for the crack-front in the element just below the surface at a depth $\sim c/100$. Again, we see that there is a maximum in the normalised stress intensity factor. For $d/c = 0{\cdot}5$ the maximum value is ~0·0185 at $c/a = 0{\cdot}16$, $r/a = 1{\cdot}2$; for $d/c = 1{\cdot}0$ the maximum value is ~0·0195 at $c/a = 0{\cdot}14$, $r/a = 1{\cdot}2$ while for $d/c = 2{\cdot}0$ the maximum value is ~0·0181 for $c/a = 0{\cdot}08$ at $r/a = 1{\cdot}1$. These values for the maxima are much closer to those obtained from the 2-D solution.

4 Discussion

We begin by noting that there is a significant difference in the stress intensity factors at the base of the crack for the 3-D solution as compared to the 2-D case; for the crack ellipticities studied the 3-D solution gives consistently lower values. For stress intensity factors evaluated near the surface the maximum values from the 3-D solution are very similar to the maximum value from the 2-D solution, although of course the 2-D solution only considers the values at the base of the crack. This suggests that while the value of the constant C to be used in eqn (1) is acceptable, crack propagation initially occurs by extension of the pre-existing flaw close to the surface rather than extension of the deeper parts of the crack.

It is pertinent to make some comments about the stability of crack growth close to Hertzian contacts. For the 2-D case we have seen that there is a maximum in the stress intensity factor: at a fixed contact radius, a, this automatically implies that $dK_I/dc < 0$—assuming that the crack propagates perpendicular to the surface. For $r/a = 1{\cdot}189$, $c/a = 0{\cdot}044$ we find that $K_I/p_o\sqrt{\pi a} = 0.018288244$; for $r/a = 1{\cdot}189$ and for a slightly longer crack, $c/a = 0{\cdot}0441$ we have $K_I/p_o\sqrt{\pi a} = 0{\cdot}018288243$ so

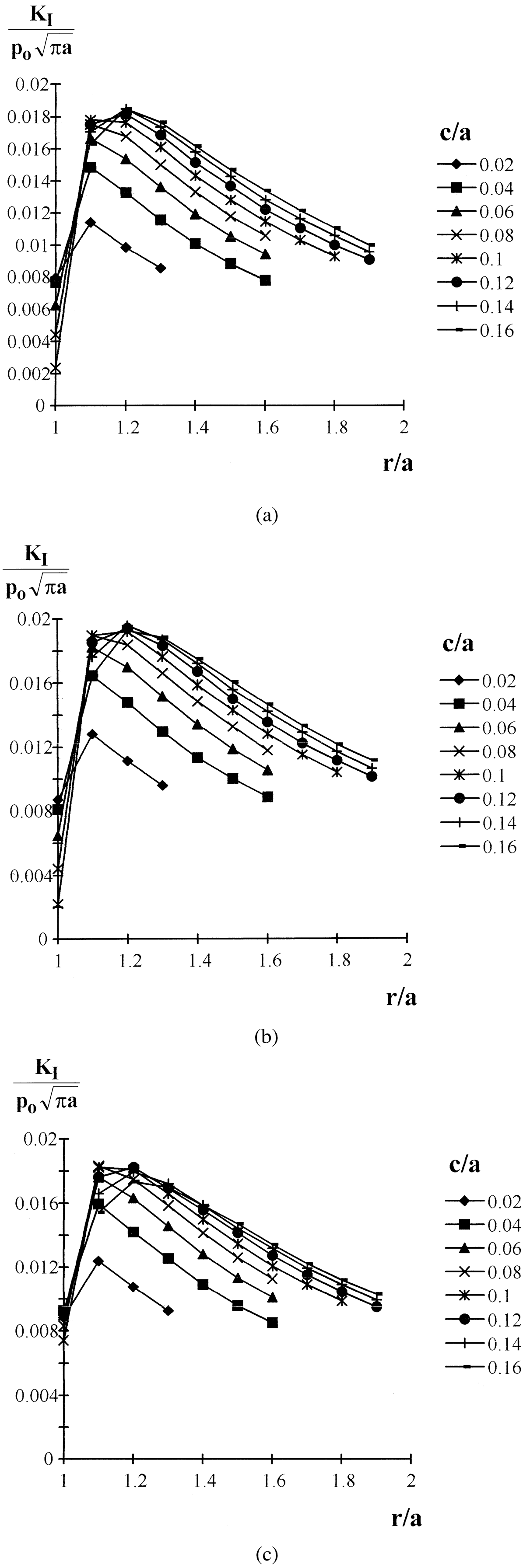

Fig. 4. Results of the 3-D calculation. Normalised stress intensity factors near the surface versus normalised crack position for various normalised crack sizes. Poisson's ratio = 0·25. (a) Ellipticity = 0·5, (b) ellipticity = 1·0, (c) ellipticity = 2·0.

that dK_I/dc is, as expected, negative. (It should be noted that we are quoting values of stress intensity factor to 9 figures simply to demonstrate that dK_I/dc is negative.) Thus, saying that a maximum in the stress intensity factor corresponds to a minimum in the fracture load is not strictly true because the requirement for fracture that $dK_I/dc > 0$ is not satisfied. Clearly, this conclusion may not be true if the crack does not propagate perpendicular to the surface: it is observed experimentally that at some depth below the surface the growing ring-crack starts to fan out, eventually giving rise to the well known cone-crack. The fracture mechanics solution for this kinked crack configuration does not, as yet, exist.

If, at the minimum load, the load is allowed to increase slightly then the calculation of the associated change in K_I becomes more complex because the values of p_o r/a and c/a all change: from eqn (4b), letting $P \to P + \delta P$ (and hence $a \to a + \delta a$) we have the following result:

$$K_I = \frac{3(P+\delta P)\sqrt{\pi c}}{\pi(a+\delta a)^2} f\left(\frac{r}{a+\delta a}, \frac{c}{a+\delta a}, \nu\right)$$
$$\delta a = \left(\frac{3R}{4E^*}\right)^{1/3} \frac{\delta P}{3P^{2/3}} \tag{6}$$

For $\nu = 0{\cdot}25$ and $\delta a = 0{\cdot}001$ we have $r/(a+\delta a) = 1{\cdot}187812$, $c/(a+\delta a) = 0{\cdot}043956$ and we obtain the new value $K_I/p_o\sqrt{\pi a} = 0{\cdot}018315653$. So, a slight increase in load leads to an increase in stress intensity factor and crack growth will occur, but the fracture is still not unstable because for $r/(a+\delta a) = 1{\cdot}187812$ and $c/(a+\delta a) = 0{\cdot}044056$ we obtain $K_I/p_o\sqrt{\pi a} = 0{\cdot}018315566$. To summarise, at the minimum load we expect crack growth to occur if the load increases slightly but not if the crack grows at a fixed load. To follow the entire 2-D crack growth process as both load and crack length change is work that is in progress. For the 3-D case the problem is even more complicated because the crack ellipticity will change as different parts of the crack propagate; as indicated above the first stage of crack growth will occur in the near-surface region, tending to increase the ellipicity.

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