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Numerical separation of bi-modal strength distributions

J. Absi, J.C. Glandus*

G.E.M.H., ENSCI, 47, avenue Albert Thomas, 87065 Limoges, France

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Abstract

This paper is devoted to the separation of bi-modal strength families. A numerical solution has been developed, because the complexity of the problem makes it unrealistic to search for an analytic solution and the bibliography has shown that the graphical method has severe limits. Specific softwares have accordingly been written which permit the processing of a whole data family and which do not require preliminary data treatment. This method have been applied to various families of strength data corresponding to increasing levels of difficulty in effecting a separation. It gives satisfactory results if the statistical parameters (Weibull modulus and mean strength) of the sub-families represented in the analysed data are not too close. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Statistical analysis; Strength; Strength distribution; Weibull modulus

1. Introduction

The strength of ceramics usually exhibits a large scattering of values (up to 100%) even for high performance structural ceramics. This well known phenomenon, which arises from the scattering of the initial sizes of the defects responsible for failure, is generally studied statisticaly by means of Weibull's analysis.[1]

However, the so-called "Weibull plot" $[\ln(\ln(1/P_s))=f(\ln(\sigma)]$ where the probability of survival, P_s, is shown as a function of the applied stress, σ, rarely exhibits the linearity characteristic of an uni-modal distribution; it is accordingly common to invoke multi-modal sources of failure. However, before accepting this hypothesis, one must verify if the non linearity can be minimised by a suitable adjustment of the distribution parameters. If this attempt fails, the problem becomes that of determining whether the sub-families can be separately identified. Though this questioning is relatively old, no fully satisfactory answer is yet available in the literature; the problem is indeed demanding since solutions are required for a four order hyperstatic analytical system.[2]

During the last 20 years, bi-modal separation techniques using graphic and/or semi-analytic methods have been published. The results are not fully satisfactory because preliminary stages are required in which it is necessary to decrease the previous hyperstaticity level (i. e. in the case of the strength of ceramics to separate samples according to their inherent flaw sizes).

Scott and Gaddiapati,[3] have built a library of ideal bi-modal curves obtained by combining two by two uni-modal Weibull distributions which have themselves been perfectly characterised. The method consists of finding, in the first instance, the curve in this library which is closest to the Weibull plot of the experimental data. The Weibull parameters of this curve constitute the starting points for an iterative calculation leading to the best "theoretical" curve, that is to say as close as possible to the real curve. These authors underlined the viewpoint that a bi-modal distribution can be invoked every time disagreement occurs between the Weibull values obtained by different methods (linear regression, maximum likelihood ...).

Estler and Bradt,[4] used fractography to identify sub-families of samples having the same mean critical size flaw (that is to say the same mean strength). This initial separation of sub-families makes it possible to overcome the problem of hyperstaticity. The unknowns become the parameters of the two Weibull sub-families built from the identified strength data. However, this identification is sometimes rather difficult and about 12% of the experimental values cannot be taken into account in the analysis.

On the basis of this work, Jakus and Ritter[5] have developed a semi-analytic method to calculate the unknown parameters.

* Corresponding author. Tel.: +33-5-5545-2222; fax: +33-5-5579-0998.

E-mail address: jc.glandus@ensci.fr (J.C. Glandus).

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Fauchon and Fraysse have proposed[6] the so-called "dynamic clouds" method. This method consists of arbitrarily choosing two quartets of consecutive values near the extremities of the global distribution, while assuming that these values belong respectively to the two unknown sub-families. Then a set of iterations is performed from these values to identify, within the real data, two families each of which being as close as possible to a uni-modal Weibull model.

Phani[7] has suggested a rewriting of the Weibull relationship:

$$\ln(\ln(1/P_s)) = m_1\ln((\sigma - \sigma_1)/\sigma_{01}) - m_2\ln((\sigma - \sigma_u)/\sigma_{02})$$

where σ_l and σ_u are respectively the lower and upper limits of the stress range.

The m_1, m_2, σ_{01} and σ_{02} parameters are then calculated by using a least mean square method.

A validation of this method using real data, has been done by Kaylan and Phani.[8]

Finally, Shariff[9] has also used a graphical method involving comparison between experimental and theoretical curves, similar in approach to the method of Scott and Gadiapolti.

To conclude, the methods available in the literature still show a lack of clarity in their methodology and validation. One can also underline that they do not take advantage of current computational power obviously needed to solve such problems.

The purpose of the present study is to develop a new separation technique based on a fully numerical method which does not require preliminary data treatment. In this aim, we have written specific programs, which are able to handle a large number of values. They have been validated by application both to simple cases and to strength data for tested structural ceramics.

The results obtained allow critical analysis of the method, its advantages and disadvantages, and an evaluation of its application conditions.

2. Analysis

Ideal "bi-modal" distributions as represented in Fig. 1 are obtained by mixing two families which individually perfectly fit the Weibull law.

On the other hand, real bi-modal distributions result from the combination of two families which have been experimentally determined to give approximate fits to the Weibull law.

In the two cases, the number of unknowns is equal to six. Indeed, for a given probability estimator (e.g. $P_s = 1-(i-0.5)/N$), the two parameter Weibull law:

$$(\ln(\ln(1/P_s)) = m\ln(\sigma) - m\ln(\sigma_0) + \ln(V)$$

requires the knowledge of the following variables:

—numbers of samples in the two families: N_1 and N_2
—mean strengths: $\sigma_{\mathbf{1}\mathrm{mean}}$ and $\sigma_{\mathbf{2}\mathrm{mean}}$
—Weibull moduli: m_1 and m_2

A knowledge of N_{total} and σ_{mean} for the total population, reduces to four the number of unknowns because of the two relationships:

$$N_{\mathrm{total}} = N_1 + N_2 \quad (1)$$

$$\frac{\sigma_{\mathrm{mean}} = (N_1^*\sigma_{1\mathrm{mean}} + N_2^*\sigma_{2\mathrm{mean}})}{(N_1 + N_2)} \quad (2)$$

In order to solve the problem analytically, four other independent relationships are needed, linking up moduli m_1 and m_2 to the other parameters. It seems very difficult (or even impossible) to find such relationships because, for a given probability estimator, both the Weibull moduli and the mean strengths depend on the number of samples.

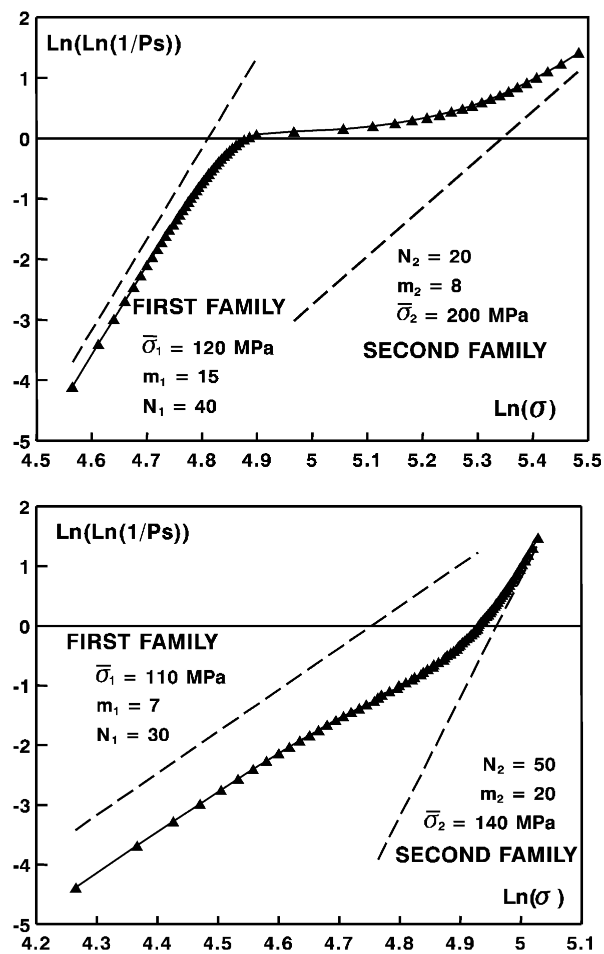


Fig. 1. Examples of ideal bimodal distributions.

Such difficulties in finding an analytical solution when combined with the problems of the graphic methods justify the development of a numerical approach.

3. Solution

3.1. Hypotheses

Whatever the distribution type (ideal or real), the present method requires that:

- the overlap of the two sub-families does not exceed 75% of the total number of data;
- each sub-family obeys a two parameter Weibull distribution ($\sigma_{u=0}$) whose values are calculated by linear regression (this work does not apply to the maximum likelihood method).

The study is divided into two parts : the first one deals with ideal families and the second with real families.

3.2. The case of an ideal families combination

The global population comprises two families each of which follows perfectly the two parameter Weibull law with regard to a probability estimator.

3.2.1. Sub-families separation principle

According to the first hypothesis, 25%, or more, of the foot values of the global distribution belong to the first family (and ditto for the head values and the second family).

So, it is sufficient to sweep foot data ($N_{foot} = N_{total}/4$), identified by their co-ordinates $\ln(\sigma_i)$ and $\ln[\ln(1/P_{si})]$, in a loop going from 1 to N, with N growing by steps of 1 between 1 and N_{foot} until the linear regression coefficient becomes maximum. The particular N_1 value thus obtained is equal to number of data for the first family of and the slope of the mean square straight line is equal to the Weibull modulus. Eventually a last sweep from 1 to N_{total} makes it possible to identify data belonging to the first family.

The same analysis applied to the head data makes it possible to characterise the second family.

A last test is performed to verify if the relationship $N_{total} = N_1 + N_2$ is well satisfied.

This process is illustrated in Fig. 2 and in Appendix A.

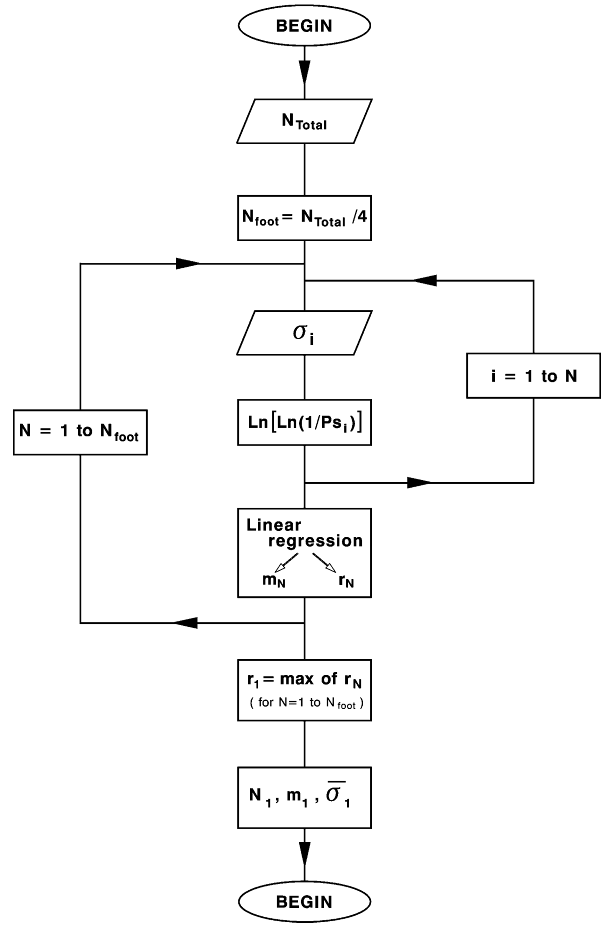


Fig. 2. Principle of separation in the case of the combination of two ideal sub-families.

3.3. The case of a real families combination

The global distribution is built, according to the method previously presented, from two sub-families each verifying the uni-modal Weibull law within a given span.

3.3.1. Complementary hypotheses

The asymptotic slopes of the Weibull plot dealing with the global distribution, noted m_{mini} and m_{maxi}, give the limits of the admissible interval of variation for the Weibull modulus of each unknown sub-family.

- $\overline{\sigma}_{1ini}$ is the mean strength of a lot of N_{1ini} values belonging to the global distribution and such as:
- the range of variation of $\overline{\sigma}_{1ini}$ covers between 80 and 90% of the global distribution and is centered on this global distribution;
- the N_{1ini} values range from $0.25{*}N_{Tot}$ to $0.75{*}N_{Tot}$.

3.3.2. Method

The separation is performed in two stages.

3.3.2.1. Stage 1. For a value m_{1ini} lying between m_{minin} and m_{maxi}, one constructs a strength family (N_{1ini} and $\overline{\sigma}_{1ini}$) verifying the uni-modal Weibull law and one verifies if all the values of this family intercept, within a given span, some data of the global population. Two cases can be encountered.

- if a full concordance is observed, the intercepted values are exclude of the global distribution and one spends to the second stage;
- if a partial concordance is observed, as long as possible one increments one of the three parameters of the studied uni-modal Weibull sub-family (N_{1ini},$\overline{\sigma}_{1ini}$ or m_{1ini}) and one reiterates the previous calculation from the beginning.

3.3.2.2. Stage 2. Once the strength of the first family are excluded of the global distribution, one verifies if the remaining values agree, with the same tolerance as in the stage 1, with the uni-modal Weibull law. If this verification is satisfied, the separation is achieved; in the opposite case, it becomes necessary to return to the first stage, as long as it remains possible to increment m_{1ini}. If such a return is impossible, one concludes to the failure of the separation.

These two stages are illustrated in the Fig. 3 and in Appendix B.

4. Application

Three types of application have been performed:

- analyse of ideal bi-modal distributions built from

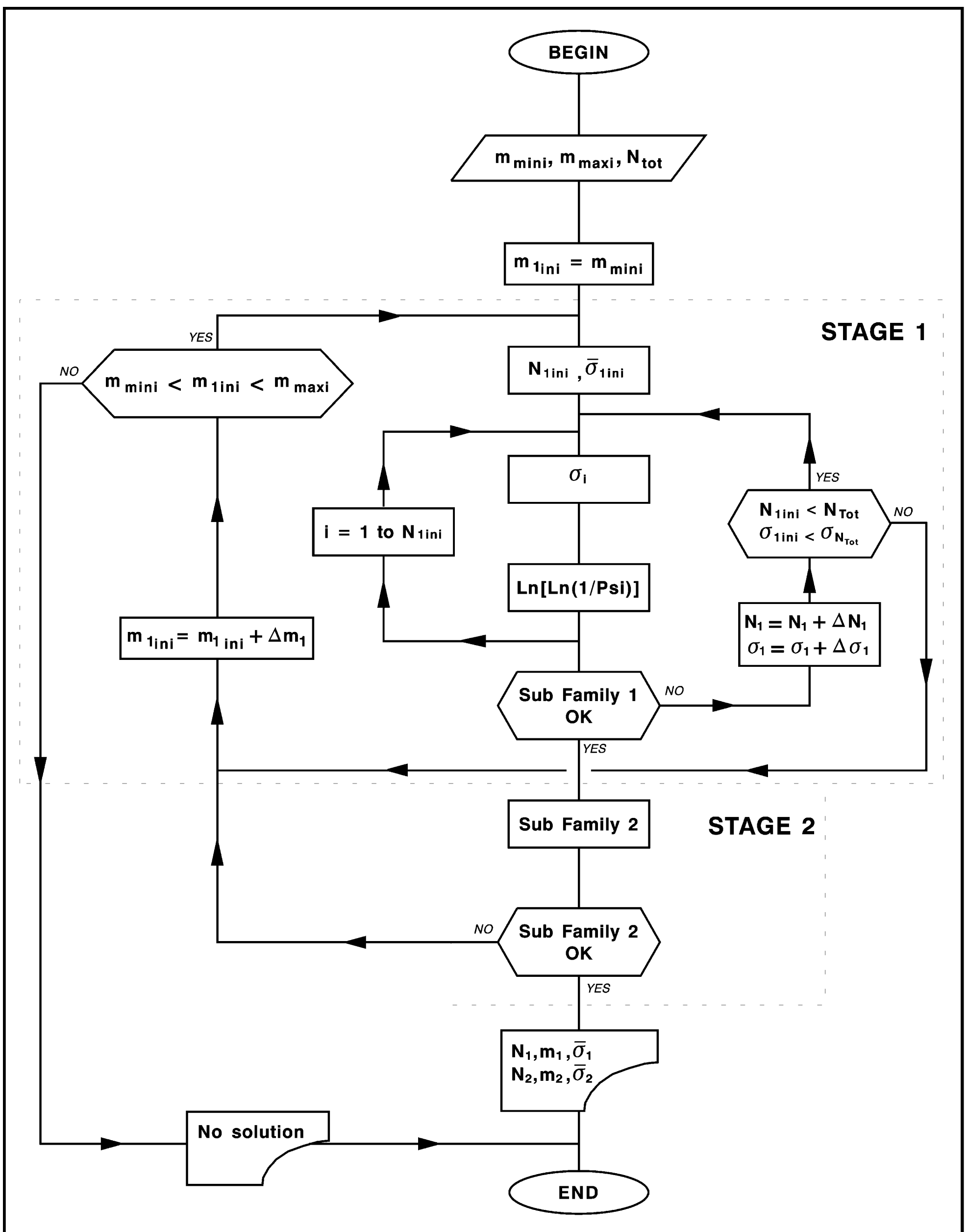


Fig. 3. Principle of separation in the case of the combination of two real sub-families.

two model families which follow perfectly the Weibull law;
- analyse of approximate bi-modal distributions built from two model families which follow roughly the Weibull law;
- analyse of real strength distributions which do not follow an uni-modal Weibull's law and for which a bi-modality of pre-existent flaws may justify the non linearity of $\ln(\ln(1/P_s))$ vs $(\ln(\sigma))$.

4.1. Combination of two families each respecting perfectly the uni-modal Weibull law

Table 1 summarises results in ten cases of separation of bi-modal families agreeing perfectly with the uni-modal Weibull law and whose compositions where previously known.

One notes a good general concordance between the parameters of the families and the results of the numerical separation except when the mean stresses are close. In this case, one observes some disagreement in Weibull modulus values and the number of samples.

These disagreements increase with the test severity and the separation becomes impossible when the mean stresses is less than 10 MPa and (or) when the number of samples in one family becomes less than 30.

4.2. Combination of two families each respecting roughly the uni-modal Weibull law

In order to validate the method in more complex cases, we have studied combinations of model sub-families which agree roughly with the Weibull law.

The results thus obtained (cf. Table 2) are broadly satisfactory, but divergences on the number of samples are always observed.

As in the previous case, these divergences increase as the separation conditions become more severe.

4.3. Real strength distributions

Real strength distributions never follow perfectly the uni-modal Weibull law and bi-modal flaw distributions are often invoked to account for this behaviour. The present method has been applied to such distributions.

4.3.1. Results of Estler and Bradt[4]

These authors measured the 4 point bending strength of 50 SiC bars ($2.8\times2.8\times38$ mm^3). Fractographic observations show that two types of flaws were responsible for failure: surface flaws and edge flaws. On this basis Estler and Bradt identify a sub-family of 11 samples exhibiting defects on their tensile face and another sub-family of 33 samples exhibiting edge defects. The 6 remaining samples exhibited defects which could not be clearly identified.

Once the numbers of samples in each sub-family were known, the other statistical parameters could be determined by an analytical method. Such an approach, involving both experimental and analytic methods, can be qualified as 'hybrid'.

By applying the present numerical method to the same data one obtains two sub-families each of which respects approximately the uni-modal Weibull law.

Results are given in Fig. 4, which shows:

Table 1
Separations of sub families respecting perfectly the unimodal Weibull's law

Data	N_1	m_1	σ_1	N_2	m_2	σ_2
Actual	40	15	120	30	11	160
Calculated	40	15	120	30	11	160
Actual	50	6	100	60	10	180
Calculated	50	6	100	60	10	180
Actual	20	18	200	90	12	210
Calculated		No solution			No solution	
Actual	120	20	175	80	14	140
Calculated	116	20.3	175	81	14	140
Actual	50	6	100	120	20	175
Calculated	50	6	100	101	21.7	177
Actual	120	20	175	150	9	130
Calculated	120	20	175	146	9	130
Actual	20	18	200	80	14	140
Calculated	20	18	200	146	9	130
Actual	30	11	160	50	6	10
Calculated	30	11	160	50	6	10
Actual	150	9	130	80	14	140
Calculated		No solution			No solution	
Actual	120	20	175	90	12	210
Calculated	15	24	154	90	12	210

Table 2
Separations of sub families respecting roughly the unimodal Weibull's law

Data	N_1	m_1	σ_1	N_2	m_2	σ_2
Actual	40	15	120	30	11	160
Calculated	40	14.9	118.5	30	11	160
Actual	50	6	100	60	10	180
Calculated	50	5.98	98.7	60	10	180
Actual	20	18	200	90	12	210
Calculated	27	18	204	83	11.5	206.5
Actual	120	20	175	80	14	140
Calculated	120	19.9	173.5	80	13.9	140
Actual	50	6	100	120	20	175
Calculated	50	6	100	120	19.9	173.5
Actual	120	20	175	150	9	130
Calculated	120	19.7	174	150	9	129
Actual	20	18	200	80	14	140
Calculated	20	17.5	198.5	80	14	140
Actual	30	11	160	50	6	10
Calculated	30	10.9	158	50	6	10
Actual	150	9	130	80	14	140
Calculated	170	8.9	130	60	14	137
Actual	120	20	175	90	12	210
Calculated	120	19.8	174	90	12	210

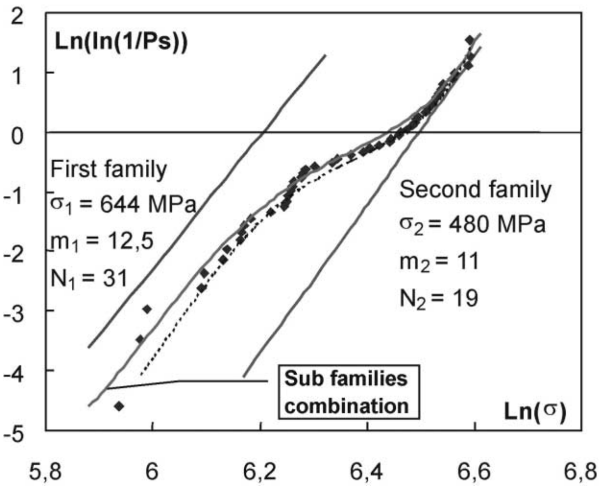


Fig. 4. Separation of Estlers and Bradt's strength data.

- the global distribution of experimental data represented by points;
- the combination of the two sub-families obtained by Estler and Bradt represented by a dotted line;
- the Weibull plots for the composing sub-families found by the present method (straight lines);
- the combination of these two sub-families represented by a continuous line.

One observes that the experimental points are well fitted by this last curve, indicating the quality of the separation achieved.

The results obtained by the two methods are summarised in Table 3 which shows some disagreement. It can be thought that this arises from the fact that the numerical method takes all values into account, whereas the hybrid one disregards about 12% of them.

Despite this disagreement, one observes in Fig. 4 that the plot obtained by combining the Estler and Bradt sub-families also gives a rather good fit to the experimental data.

So, it can be though that it does not exist a unique solution to separate a lot of real strength data into two sub-families agreeing with the uni-modal Weibull law.

4.3.2. Results of Scott and Gaddiapati[3]

The tensile strength of 119 silica fibres (125–129 μm in diameter) was measured at 2000 °C. By using the method presented in Section 1 above, these authors found two sub-families each of which exhibiting the typical low Weibull moduli of glass materials.

Applied to the same data, the present numerical method makes it possible to identify two sub-families with significantly different caracteristics (cf. Table 4). However, their combination leads to a Weibull plot which well fits the experimental points, as shown by Fig. 5.

4.3.3. Results of Arone[10] and Brousse[11]

Arone studied the strength of 26 RBSN bars (3×4×35 mm^3) tested in 3 point bending (lot a) and that of 30 RBSN bars loaded in 4 point bending (lot b).

Brousse studied the 3 point bending strength of 84 bars (4×4×38 mm^3) of a 98% pure alumina loaded at 0.5 mm/mn and that of 36 bars of the same material loaded at 0.05 mm/mn.

Contrary to the previous instance, these authors did not consider the possible bi-modality of preexistant flaws in the tested samples. However, the Weibull plots of their results are sufficiently far from straight lines to allow consideration of two composing sub-families. The results then obtained by the numerical method are illustrated by Figs. 6 (Arone) and 7 (Brousse) *but they cannot be compared to other proposals.*

Fig. 6a shows a good fitting of experimental points by the calculated curve, except for the 2 points in the foot and the 3 points in the head of the global distribution. In the same manner, in Fig. 6b, the calculated curve well fits the real distribution.

Table 3
Comparison of numerical and hybrid methods results

Method	Family 1			Family 2		
	N	m	σ_{mean} (MPa)	N	m	σ_{mean} (MPa)
Hybrid	33	7.6	553	11	10.5	671
Numerical	31	11	644	19	7.6	480

Table 4
Comparison of numerical and graphical methods results

Method	Family 1			Family 2		
	N	M	σ_{mean} (MPa)	N	m	σ_{mean} (MPa)
Graphical	80	4.7	1030	39	2.05	1910
Numerical	98	2.7	1373	21	18	789

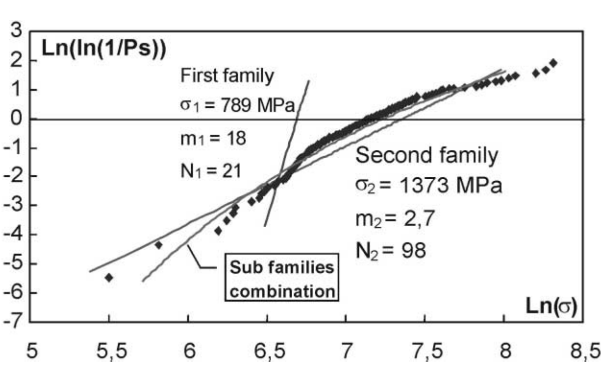


Fig. 5. Separation of Scott and Gadiopolti's strength data.

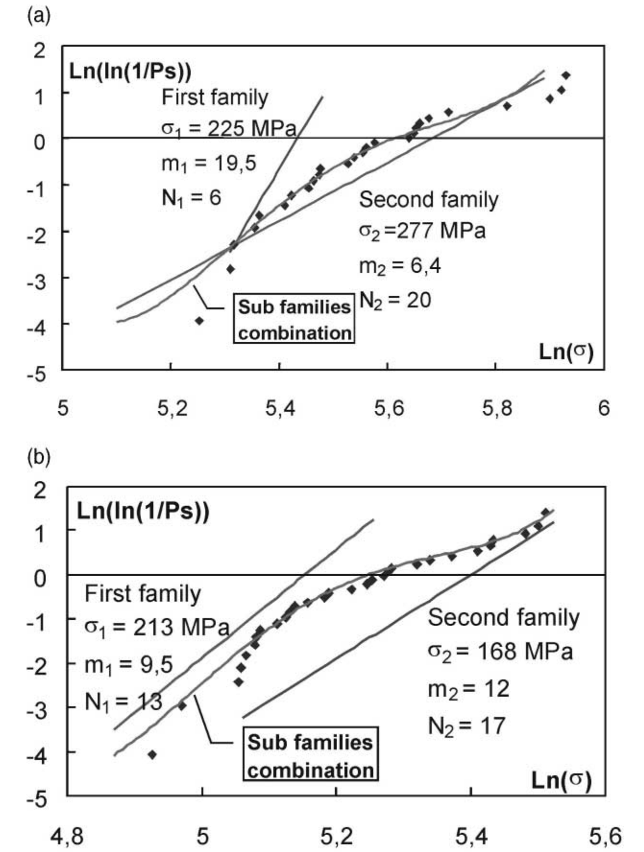


Fig. 6. Separation of Arone's strength data (a) 3 point bending; (b) 4 point bending.

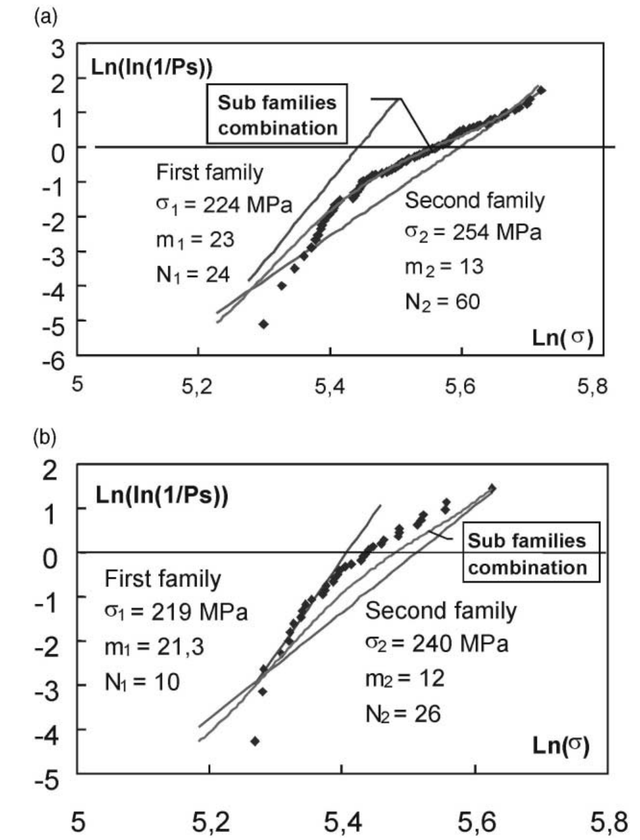


Fig. 7. Separation of Brousse's strength data (a) 0.5 mm/mn; (b) 0.05 mm/mn.

Brousse's results are illustrated by Fig. 7a and b. Fig. 7a shows that the foot values of the calculated curve disagree with the experimental data. However, this disagreement deals with 7 values out of 84, i.e. 8% only of the global population. So, it can be considered that the numerical separation gives acceptable results, because more than 90% of the strength data are perfectly separated. Finally, Fig. 7b is a good example of total disagreement between the numerical separation results and the experimental data.

5. Discussion

Applied to cases of increasing difficulty, the present numerical separation method shows broadly satisfactory results.

In cases of sub-families following perfectly the uni-modal Weibull law, it leads to very good results if the characteristics of the constituent sub-families are not too close. In the case of real distributions, the results remain satisfactory, even if the numerical method can exhibit some weakness (Figs. 5 and 7a) or, even, a total incapacity to identify reliable sub-families (Fig. 7b).

6. Conclusion

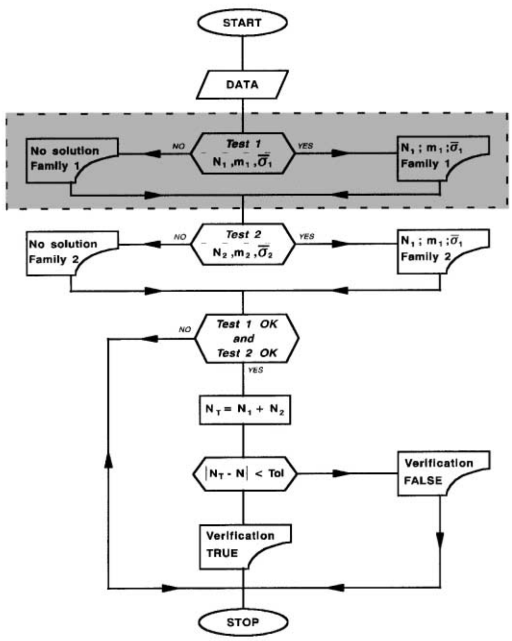
A fully numerical method has been developed to separate the sub-families in bi-modal strength distributions. It can be used in many cases and does not require any preliminary data treatment (graphical or analytical). Its validity has been proved on model bi-modal distributions, obtained by combining sub-families following perfectly or approximately the uni-modal Weibull law. Its performances were also validated in the case of real strength distributions. Moreover, it does not call for the use of a powerful computer.

However, in some cases (sub-families having low numbers of samples or exhibiting comparable mean stresses), it is unable to separate the constituent data.

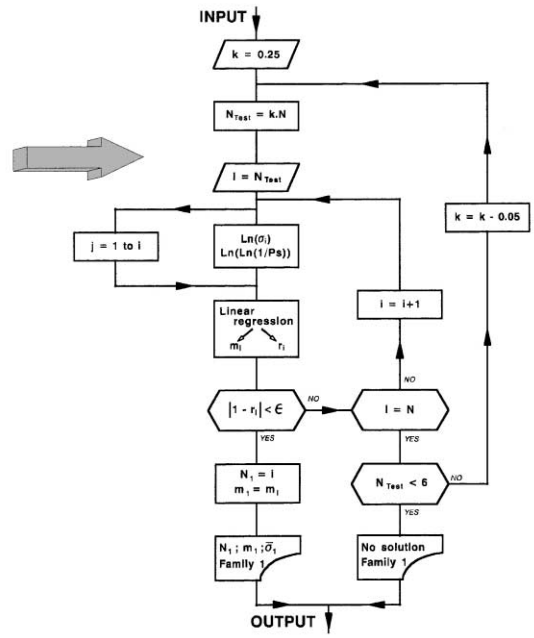
This behaviour confirms the difficulty of separating reliably the sub-families in a bi-modal distribution and calls for the further improvement of the algorithms developed for this work. Indeed, a complete work would deal with strength data involving more than two sub-families, but the level of difficulty obviously increases considerably with the number of sub-families. This fact explains why, up to day, only bi-modal populations have been studied.

Nevertheless, before performing such improvements it must be clearly establish if the family obtained by combining two real strength data admits a single or many solutions.

Appendix A

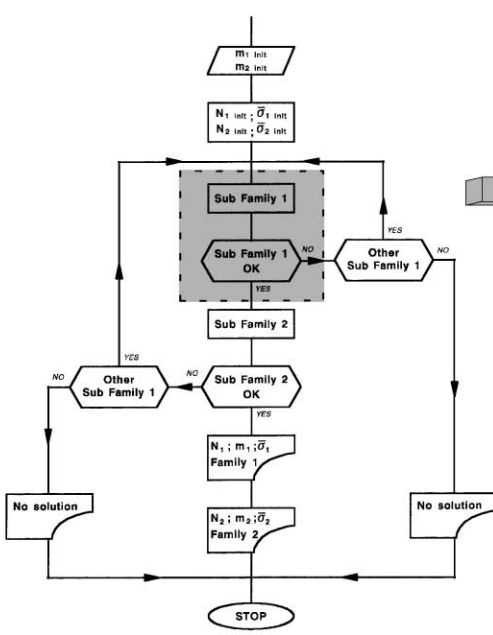


Global algorithm



Detail of the algorithm :
research of the first family roots

Appendix B



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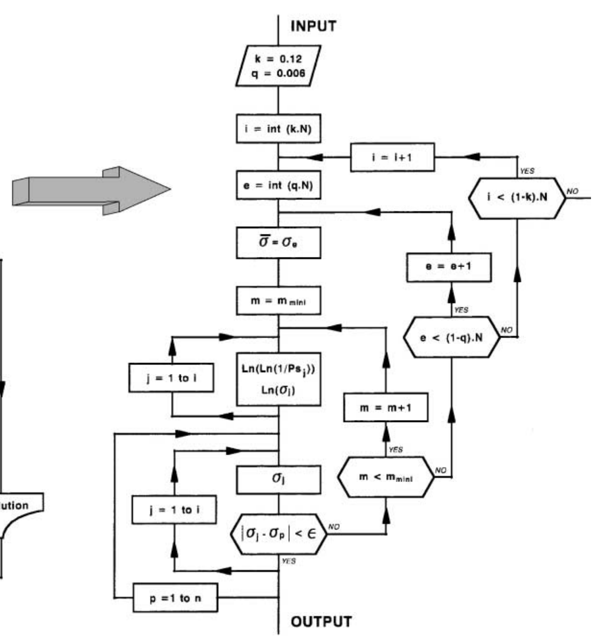

Detail of the algorithm :
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