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Some notes on the correlation between fracture and defect statistics: Are Weibull statistics valid for very small specimens?

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Abstract

Strength data of brittle materials show a significant scatter. Therefore probabilistic methods have to be used for designing with these materials. So far this has been done on the basis of the Weibull statistics. The Weibull statistics (implicitly) implies a particular type of defect distribution, which can be observed in many (but not in all) ceramic materials. The correlation between the strength and the flaw size distribution is discussed in some simple examples. Then the situation for very small specimens is discussed. This case will be of high relevance for the testing of miniaturised electroceramic components and materials of the microsystem technique. It is shown that the Weibull theory gets inconsistent and should overestimate the strength of the specimens (components), due to the fact that the effective volume becomes smaller than the fracture origin.

Keywords: Strength; Defects; Fracture; Weibull statistics

1. Introduction

Fracture of brittle materials (e.g. ceramics) usually initiates from flaws, which are distributed in the material. The strength of the specimen then depends on the length of the major flaw, which varies from specimen to specimen. Therefore, the strength of brittle materials has to be described by a probability function (statistics).[1–3] It follows from the experiments that the probability of failure increases with load amplitude and size of the specimens.[1,2,4] The first observation is trivial. The second observation follows from the fact that it is more likely to find a major flaw in a large than in a small specimen. Therefore, the mean strength of a set of large specimens is smaller than the mean strength of a set of small specimens. This size effect of strength is the most prominent and relevant consequence of the statistical behaviour of the strength of brittle materials.

Weibull was the first to develop a statistical theory of brittle fracture.[5,6] His fundamental assumption was the weakest link hypothesis, i.e. the specimen fails if its weakest volume element fails. Using some empirical arguments necessary to make a simple and good fitting of his experimental data he derived the so called Weibull distribution of the probability of failure, F, which, in its simplest form and for a uniaxial homogenous and tensile stress state, σ, and for specimens of the volume, V, is given by:

$$F(\sigma, V) = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right]. \quad (1)$$

The Weibull modulus, m, is a measure for the scatter of strength data: the smaller the m is, the wider the distribution is. σ_0 is a characteristic strength value and V_0 the corresponding reference volume. For $V = V_0$ and $\sigma = \sigma_0$, the probability of failure is equal to $1-e^{-1} \cong 63\%$. Of course the probability of surviving (the reliability R) is given by $R = 1-F$.

In the last 50 years a significant amount of research has been made to give Weibull's theory a more fundamental basis.[7–15] The paper of Kittl and Diaz[16] gives a good overview on the former developments. Freudenthal[8] showed for homogenous and brittle materials that, if the flaws do not interact (i.e. if they are sparsely distributed), the probability of failure only depends on the number of destructive flaws, $N_{c,S}$, occurring in a specimen of size and shape, S,

$$F_S(\sigma) = 1 - \exp\left[-N_{c,S}(\sigma)\right]. \quad (2)$$

$N_{c,S}$ denotes the mean number of destructive (critical) flaws in a large set of specimens (i.e. the value of expectation). Jayatilaka

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et al.[9] demonstrated in their noteworthy paper, that, for a brittle and homogeneous material, the distribution of the strength data is caused by the distribution of sizes (and orientations) of the flaws and that a Weibull distribution of strength will be observed for flaw populations with a monotonically decreasing density of flaw sizes. Danzer et al.[12–14] extended these ideas to flaw populations with any size distribution and to specimens with an inhomogeneous flaw population. Then the strength distribution strongly depends on the shape of the flaw distributions in the material. Again it was necessary to assume that a specimen fails if any one flaw initiates fracture (the weakest link hypothesis), and that there is no interaction between the flaws. On the basis of these ideas a direct correlation between the flaw size distribution and the scatter (statistics) of strength data can be recognised.

The failures-causing flaws are in most cases strictly related to the materials production process. To give some examples—such flaws are large grains, pores resulting from organic inclusions, pressing defects ore remnants of agglomerates.[17] It seems to be obvious that size and distribution of such flaws strictly depend on the production process and they change with size and geometry of the component. Therefore, it is of great importance to determine the mechanical properties separately for each type of component. This has to be done on specimens, which are machined out of the components or on the components themselves.

Up to these days the Weibull distribution function has been the basis of the state of the art mechanical design process of ceramic components.[18,19] The strength testing of ceramics and the determination of Weibull distribution are standardised.[20,21] Following the standards the Weibull distribution function has to be measured on a sample of "at least" 30 specimens (due to high machining costs, larger samples are hardly tested in the daily testing practice). It should be recognised that the range of "measured" failure probabilities increases with the sample size[14] and is – for a sample of 30 specimens – very limited (it is between $\sim 1/60$ and $\sim 59/60$). To determine the design stress, the measured data have – in general – to be extrapolated with respect to the (effective) volume and to the "tolerated" failure probability of the components, respectively, which often results in a very large extrapolation span.[22] In order to reduce this extrapolation span the testing of large samples and of specimens with a size as large as possible has been recommended.[22]

Electroceramic components have a large and growing market nowadays and are produced in gigantic numbers (to give an example: the annual world production of PTC components reaches several billions of components). It is well known that, in service, the Joule self-heating of electroceramic components may cause large temperature changes and differences in the components. This may cause serious mechanical stress and even mechanical failing of the components.[23–25] But, in general, a very high service reliability (of around 99.9999% = $1–10^{-6}$ and more) is claimed for electroceramic components. Therefore, mechanical testing and mechanical reliability assessment of such components is a relevant topic,[26–29] which should be more addressed. Since electroceramic components are very small in general (much smaller than the standardised size for bending test specimens[20]) and since the miniaturisation of these components still proceeds, challenging work on appropriate mechanical testing procedures for small electroceramic components (or for small specimens machined out of the components) must be done.[30,31]

In this paper the application of the Weibull theory on very small specimens is reviewed. The relationships between flaw size distribution, the size of the fracture initiating flaw and the strength are discussed, and it is shown that there exists a limit for the applicability of the classical fracture statistics (i.e. Weibull statistics based on the weakest link hypothesis) for very small specimens (components).

2. Fracture statistics and defect size distribution

To determine the probability of failure in dependence of the applied load more information on the involved flaw distributions is needed.[9,12–14] The function $N_{c,S}(\sigma)$ is obtained by integrating the local density, $n_c(\sigma, \vec{r})$, of destructive flaws:

$$n_c(\sigma, \vec{r}) = \int_{a_c(\sigma)}^{\infty} g(a, \vec{r})\,\mathrm{d}a \tag{3}$$

over the volume of the specimen:

$$N_{c,S} = \int n_c\,\mathrm{d}V. \tag{4}$$

For simplicity and without loss of generality[9] it has been assumed that the size and orientation of a flaw can be described by the single variable (the flaw size, a). The frequency distribution density of flaw sizes, $g(a, \vec{r})$, may depend on the position vector, $\vec{r}$. A local fracture criterion (e.g. the Griffith criterion,[1,2]) correlates stress amplitude and flaw length: the critical flaw size, $a_c(\sigma)$, is the minimum flaw length, which – under the action of the stress σ – causes failure (the size of the smallest destructive flaw). Since a_c depends on the magnitude of the applied stress, so do the values of n_c and also of $N_{c,S}$.

It has been shown by Jayatilaka et al[9] that for homogeneous materials loaded under uniaxial homogeneous tension (in this case the integral in Eq. (4) is trivial) and for flaw populations with relative frequencies that decrease according to a negative power of their (effective) radius, a,

$$g(a) = g_0\left(\frac{a}{a_0}\right)^{-r}, \tag{5}$$

a Weibull distribution (Eq. (1)) occurs. Of course it is still assumed that the flaws do not interact. This function has only two parameters effectively: the exponent $(-r)$ and a coefficient $(g_0 a_0{}^r)$. The density of destructive flaws in terms of a critical flaw size a_c is:

$$n(a_c) = \int_{a_c}^{\infty} g(a)\,\mathrm{d}a = \frac{g_0 a_0{}^r}{r-1} a_c^{1-r} = \frac{a_c}{r-1} g(a_c). \tag{6}$$

Inserting Eq. (5) into Eq. (6), performing the integral of Eq. (4) and applying the Griffith criterion[1,2]

$$a_c = \frac{1}{\pi}\left(\frac{K_{Ic}}{Y\sigma}\right)^2, \tag{7}$$

which relates the critical flaw size a_c to the tensile strength σ, the Weibull distribution, Eq. (1), results. K_{Ic} is the critical stress intensity factor (the fracture toughness) and Y is a dimensionless geometric factor. The Weibull parameters are given by

$$m = 2(r - 1), \tag{8a}$$

and

$$V_0{\sigma_0}^m = \frac{r-1}{g_0 a_0}\left(\frac{K_{Ic}}{Y\sqrt{\pi a_0}}\right)^m. \tag{8b}$$

In the following, a homogeneous material behaving in the way described above (Eqs. (1) and (5), etc.) is called "Weibull material".

It should be noted that a Weibull-type strength distribution (Eq. (1)) also may arise for inhomogeneous stress and non-uniaxial stress states (then the volume has to be replaced by an effective volume[1]). If failure is caused by surface flaws, the volume in Eq. (1) has to be replaced by the surface.[1,13]

In a series of papers Danzer et al. have discussed the influence of several other flaw size distributions as given in Eq. (5) (e.g. of bimodal distributions) on the strength distribution.[13,14,22] In these cases the Weibull modulus migth depend on the applied load amplitude and on the size of the specimen. Then the determination of a design stress in the usual way may become problematic. A stress- and size-dependent modulus may also be caused by internal stress fields[13] or occurs for materials with an R-curve behaviour.[14]

It should also be noted that on the basis of a small sample size, e.g. containing 30 specimens, it is not possible to differentiate between a Weibull, a Gaussian, or any other similar distribution function. This has been shown by means of chi-square tests and similar tests in a paper by Lu et al.[32] on the example of experimentally determined distributions and in a paper by Danzer et al.[14] by Monte Carlo simulations. In other words, if a strength distribution is evaluated according to the rules of the standards, e.g.,[21] it appears to be a Weibull distribution in almost any case. This is a consequence of the inherent scatter of the data and of the sampling procedure for small samples. The ultimative test for the existence of a Weibull distribution is to test a material on different levels of (effective) volumes.

3. Determination of the flaw population from fracture tests

3.1. Tensile tests

Let us first discuss the relationship between fracture statistics and defect size distributions for the simple case of tensile tests (uniaxial and homogeneous stress state) of a homogeneous brittle material. The tests are performed on specimens of equal size. For reasons of convenience it is assumed that the specimens volume is $V = V_0$. The number of tested specimens is X (i.e. the sample size is X). In each test the load is increased up to the moment of failing. The strength is the stress at the moment of failure. In each sample the strength values of the individual specimens are different, i.e. the strength is distributed.

If the data determined in that way are evaluated the specimens are ranked according to their strength, i being the ranking parameter. To estimate the failure probability for an individual specimen an estimation function is used[#1]:[1,21]

$$F_i = \frac{i - 1/2}{X}, \quad i = 1, 2, \ldots, X. \tag{9}$$

Inserting Eq. (9) into Eq. (2) and making a few rearrangements result in:

$$N_{c,S}(\sigma_i) = \ln \frac{2X}{2X - 2i + 1}. \tag{10}$$

In this way the mean number of destructive flaws per specimen (of volume V_0) at the stress σ_i can be read by the ranking number and sample size.

For the weakest specimen ($i = 1$) of a sample (size X) the estimator for the probability of failing is: $F_1 = F(\sigma_1) = 1/2X$. That specimen contains in the mean $N_{c,S}(\sigma_1) = \ln 2[X/(2X - 1)]$ destructive flaws. For the strongest specimen of the sample ($i = X$) it holds: $F_X = F(\sigma_X) = (2X-1)/2X$ and $N_{c,S}(\sigma_X) = \ln 2X$.

A special situation occurs if the strength is equal to the characteristic strength (i.e. for $V = V_0$ and $\sigma = \sigma_0$). Then the probability of failure is $F(\sigma_0) = 1 - 1/e$ and the mean number of destructive flaws per specimen is equal to one: $N_{c,S}(\sigma_0) = 1$. Therefore, for homogeneous distributed flaws, the density of destructive flaws is:

$$n_c(\sigma_0, V_0) = \frac{N_{c,S}(\sigma_0)}{V_0} = \frac{1}{V_0}. \tag{11}$$

For a Weibull material inserting Eq. (11) in Eq. (6) and rearranging the result for the relative frequency distribution of flaw sizes is:

$$g(a_{c,0}) = \frac{r-1}{V_0 a_{c,0}}. \tag{12}$$

If the calculations made for $\sigma = \sigma_0$ and $V = V_0$ are generalized for any stress value σ_i, and for specimens of any volume V the equation reads:

$$g(a_{c,i}) = \frac{r-1}{V a_{c,i}} \ln \frac{2X}{2X - 2i + 1}. \tag{13}$$

where $a_{c,i}$ is related to σ_i via Eq. (7). The application of this equation opens a simple possibility to determine the frequency distribution of flaw sizes in a wide range of parameters by testing specimens of different volume.

[#1] There exists a wide literature on the best definition of the estimation function and – although in most practical cases the differences between the proposals are small (compared to the scatter of data) – there is no consensus about the best choice of that function. The aim of this paper is to make clear the relationship between strength and flaw size distribution and to show the implicit limits of the applicability of the Weibull theory. Therefore, small differences in the definition of the estimation function are not relevant and the simple function of Eq. (9) is used.

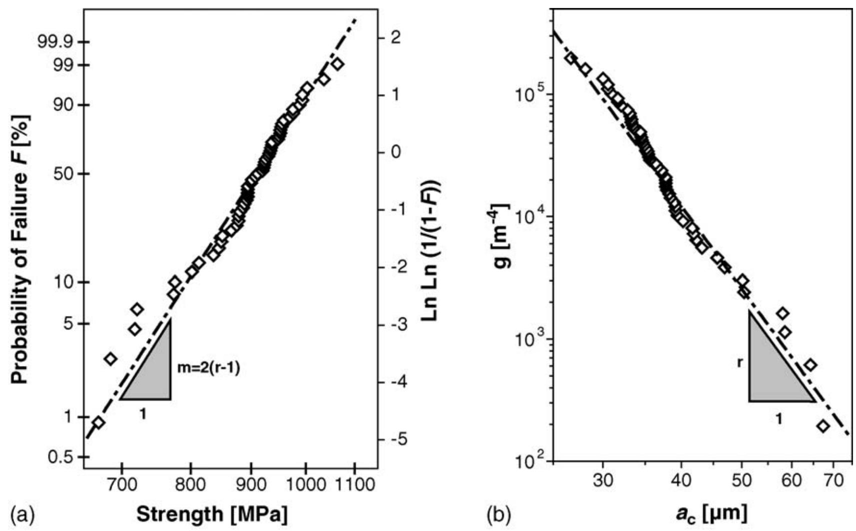

Fig. 1. Strength data of a silicon nitride ceramic tested in four-point bending[14,33]: (a) Weibull diagram (probability of failure vs. stress); (b) frequency distribution density of flaw sizes vs. the flaw size in a double logarithmic representation.

3.2. Determination of the relative frequency of flaw sizes from strength tests; bending tests on a silicon nitride ceramic

A large set of bending test results[#2] is published in [14,33]. Fifty-five bending bars were machined out of the ceramic valves made of a pressureless sintered silicon nitride material. The specimen size was 3 mm × 3 mm × 35 mm. To avoid the surface flaws caused by the machining of the specimens their tensile surfaces were carefully finished according to ENV 843-1.[20] The tests were performed using a fully articulating fixture with a 30/15 roller span in analogy to ENV 843-1.[20] These data are used in the following to visualize the relationship between the Weibull and the flaw size statistics. The weakest, the characteristic, and the strongest strength value of the samples were determined to be 669, 933 and 1064 MPa, respectively, the Weibull modulus of the sample is $m = 14$ and fracture toughness[#3] is $K_{Ic} \approx 6.2\,\mathrm{MPa\,m^{1/2}}$.[33] Using Eq. (7) and setting $Y = 2/\pi$ (the geometric factor of the penny-shaped crack) the critical flaw sizes according to the above strength data are ∼67, ∼37, ∼26 μm, respectively. The size ratio of the critical flaws in the weakest and the strongest specimen is around 2.6. For the tested silicon nitride, all the analyzed fracture origins were yttrium and aluminum reach agglomerates.[33]

The results of the strength tests are plotted in a Weibull diagram[1] (Fig. 1a). The data are nicely grouped around a straight line, which represents the Weibull distribution calculated by the maximum liklihood method according to ENV 843-5.[22] The slope of the line is the Weibull modulus ($m = 14$). An evaluation of the data according to Eq. (13)[33] results in Fig. 1b, where the relative frequency of flaw sizes is printed versus the critical flaw size in a double logarithmic plot. Again the data are nicely grouped around a straight line (now it has the slope $-r$, with $r = 8$). An extremely strong dependency of the relative frequency on the critical flaw size can clearly be recognized: although the size of the critical flaws in the samples only varies by a factor of around 2.6 the corresponding relative frequency of the flaw sizes varies by a factor of around 2000 ($2.6^8 \approx 2000$). That means that agglomerates with a radius of 26 μm occur about 2000 times more often than agglomerates with a radius of 67 μm.

In the past, inhomogeneities in the microstructure have been identified many times to be the origins of fracture in brittle materials.[17] In general, it can be concluded from fracture experiments that the Griffith criterion applies (Eq. (7)) quite well,[17,35–37] and nowadays it is the fundamental idea and basic guide line in the standards for fractography, e.g. [38].

3.3. Relationship between density of destructive flaws, the number of destructive flaws per specimen, the failure probability, the strength and size of the critical flaws for a typical Weibull material

In this example, parameters of a typical Weibull material are used which could correspond to the measured data of a modern silicon carbide or of a modern alumina ceramic. For that material the insinuated correlation between strength and defect statistics is studied in more detail. The following parameters assumed for that material are[#4]: $\sigma_0 = 400\,\mathrm{MPa}$, $m = 20$ (determined at specimens with $V = 1\,\mathrm{mm}^3$) and $K_{Ic} = 4\,\mathrm{MPa\,m^{1/2}}$. The geometric factor of the defects is chosen to be equal to that of

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[#2] To make the discussion easier the arguments for tensile tests are made (specimens tested with an uniaxial and homogene tensile stress state) but the ideas can also be applied to other testing conditions. In the case of bending tests, the procedure is described in standard text books, e.g.[1,2] The specimens volume has to be replaced by an effective volume and the tensile stress by an reference stress (the outer fibre stress).

[#3] The fracture toughness has been determined in [34] using the indentation strength and indentation fracture method. Both methods have resulted in almost the same value.

[#4] For many modern ceramic materials the ratio of K_{Ic}/σ_0 is about $1/100\,\mathrm{m}^{1/2}$. The Weibull modulus of most modern ceramics is between 10 and 20 and in rare cases it can be even be higher. The relative high value of $m = 20$ has been chosen to make the discussed trends more evident.

a penny-shaped crack in the volume: $Y = 2/\pi$. For this material the parameters in Eq. (5) are:

$$r = 11 \quad \text{and} \quad g_0 = \frac{4}{\pi} 10^{14} m^{-4}. \tag{14}$$

The free scaling parameters are defined to be $V_0 = 1\ \text{mm}^3$ and $a_0 = (\pi/4) \times 10^{-4}$ m. The data were selected in such a way that at $\sigma = \sigma_0 = 400$ MPa the critical crack size is $a_c = a_0$, the relative frequency of flaw sizes is $g(a_0) = g_0$ and the density of destructive flaws is $n(a_c) = 10^9\ \text{m}^4$.

For this material and in a tensile specimen with a volume of $V = 1\ \text{mm}^3$, 1 destructive flaw occurs per specimen (in the mean) and in a tensile specimen with a volume of $10\ \text{mm}^3$, 10 destructive flaws per specimen occur in the mean. In the first case the failure probability is 63% but in the latter it is $99.995\% \cong 100\%$. The difference to 100% accounts for the case that in some specimens more than the mean number of destructive flaws exist, but in some rare cases there is none (in the second example in one of 2×10^4 specimens).

At a lower stress, as in the case discussed above, say at $\sigma = 0.9\sigma_0 = 360$ MPa the critical crack length is 21% larger and the density of destructive flaws is $1.21^{(r-1)} = 1.21^{10} \cong 6.7$ times lower than at $\sigma = \sigma_0 = 400$ MPa. The corresponding probability of failure (in a specimen with a $V = 1\ \text{mm}^3$) is around 14%. At a stress level of $\sigma = 1.1\ \sigma_0 = 440$ MPa the density of destructive flaws is around 6.7 times higher than for $\sigma = \sigma_0$ and the probability of failure is 99.9%.[#5]

It is obvious that for modern ceramic materials having a high Weibull modulus there is an extremely strong dependency of the probability of failure on the frequency distribution density of flaw sizes (and on the density of destructive flaws, respectively).

The correlations between density and the number of destructive flaws, the critical flaw size, the applied stress and the probability of failure are visualized in Fig. 2. The thick line corresponding to Eq. (6) describes – for specimens having a volume of 1 mm^3 – the relationship between the number of destructive flaws and the critical flaw size (in relative units). All properties are plotted in a double logarithmic scale. The slope of the line corresponds to that of a material having a Weibull modulus of $m = 20$ (the slope is $-m/2 = 1 - r$). Using the Griffith/Irwin criterion (Eq. (7), $a_c \propto \sigma^{-2}$) each flaw size is related to a strength value. The corresponding scale for the strength (in relative units) is plotted on the top of the diagram. Using Eqs. (1) and (2) each strength value also corresponds to a probability of failure, the corresponding scale is plotted on the right side of the ordinate. Values for the density of destructive flaws are plotted on the rigth hand side of the diagram.

3.4. Size effect on strength, application to very small specimens

As discussed in the introduction there exists a size effect on strength[1,2] which – for a Weibull material – is described by:

$$V_1 {\sigma_1}^m = V_2 {\sigma_2}^m \tag{15}$$

[#5] For $\sigma = \sigma_0/2$, the density of critical flaws is $2^{(r-1)} = 2^{10} = 1024$ times the density at $\sigma = \sigma_0$ and for $\sigma = 2\sigma_0$ it is 1/1024 times the density at $\sigma = \sigma_0$.

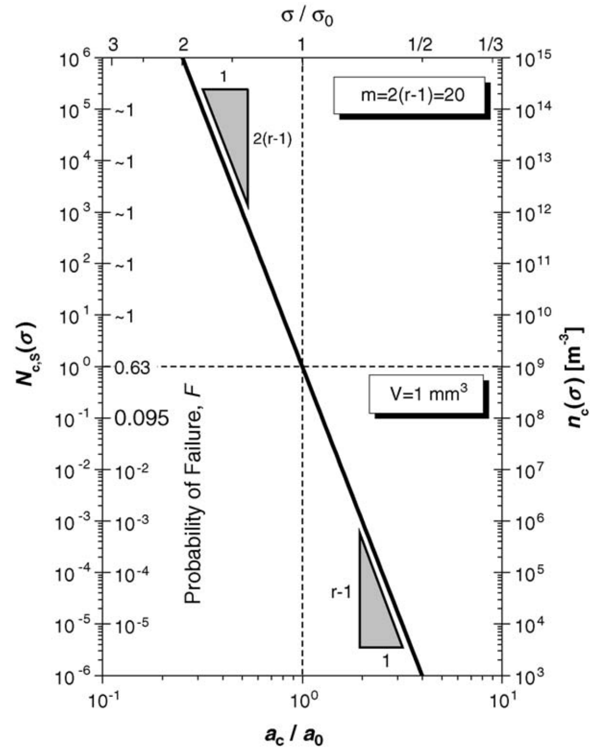

Fig. 2. Number of destructive flaws (or probability of failure) for specimens of volume $V = 1\ \text{mm}^3$ vs. the critical flaw length (in relative units) for a Weibull material having a Weibull modulus of $m = 2\ (r-1) = 20$. The relationship on the strength is also indicated (top of the diagram). The ordinate at the right hand side of the diagram shows the density of destructive flaws per specimen. For specimens having other volume (V_2) this scale is shifted upwards by the constant factor $\lg(V_2/V)$.

The probability of failure in a sample containing specimens of volume V_2 is equal to that in another sample containing specimens of volume V_1 if the stresses applied to the specimens are related according to Eq. (15). The size effect can also be included in Fig. 2: for specimens having another volume than $V = V_0 = 1\ \text{mm}^3$ the number of destructive flaws (and the probability of failure) causes a shift of the right ordinate by the constant $\lg (V_2/V_1)$ (remember, the flaw density in a Weibull material is assumed to be homogeneous).

Inserting Eq. (7) into Eq. (15) shows a relationship for the radii of the corresponding critical defects:

$$\frac{a_{c,1}}{a_{c,2}} = \left(\frac{V_1}{V_2}\right)^{2/m}. \tag{16}$$

The corresponding relative frequency of flaw sizes for sample 1 at $\sigma = \sigma_0$ is $g(a_1) = g_0(a_1/a_0)^{-r}$ and the density of destructive flaws is $n_c(\sigma_{0,1}, V_1) = 1/V_1$. The analogue holds for the sample with specimens of volume 2.

In Fig. 3[#6] the diameter ($2a_c$) of the critical flaws corresponding to the characteristic strength of the specimens is plotted in a double logarithmic scale and in relative units versus the volume of the specimens (Eq. (16)). The slope of the line is $2/m = (r-1) = 0.1$. The dashed line corresponds to a typical

[#6] A similar diagram can be found in the thesis of A. Börger.[40]

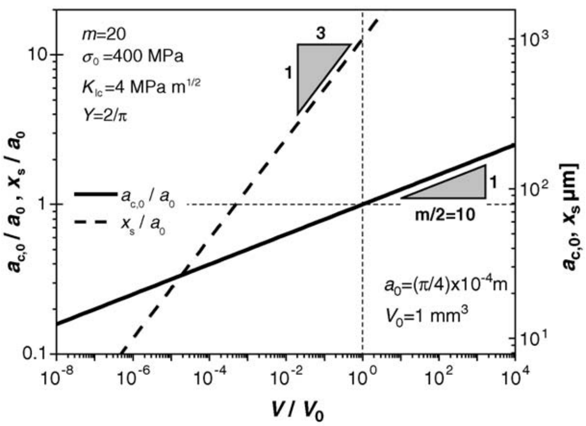

Fig. 3. Diameter of the critical flaw size ($2a_c$) vs. the volume of the specimen (full line) in a double logarithmic plot. The edge length of a cube having the volume of the specimen (dashed line) is also shown. The material properties are equal to those used in Fig. 2 ($m = 20$, $\sigma_0 = 400$ Pa, $K_c = 4$ MPa m$^{1/2}$, $V_0 = 1$ mm^3, $Y = 2/\pi$).

scaling parameter for the volume of the specimen; for simplicity the edge length of a cube with volume V is taken as the characteristic length $x_S = V^{1/3}$. For materials with a modulus of $m > 6$, there exists a point of intersection between both lines, which is – in the selected example – at a volume of about $V \approx 2 \times 10^{-4}$ mm^3 (this corresponds to the radius of the critical flaw of about $a_c \approx 30$ µm). The full line shifts down (and the intersection point to the left) for materials having a lower K_{Ic}/σ_0 ratio as used in the above example. This might happen for some materials with an advanced processing technology. The intersection point shifts to the right for a material with a higher Weibull modulus (and to the left for a materials with a lower modulus) as used in the above example ($m = 20$). In summary it can be stated that for many advanced materials (having a modulus $m = 10$–20) the intersection point will occur at an (effective) volume of the specimen of about 10^{-4} mm^3 or even at a smaller volume.

Obviously the assumption made in Eq. (5) (the relative frequency of flaws follows an inverse power law) can only approximate the behaviour of materials for large flaws and it fails for very small flaws: the relative frequency goes to infinite if the flaw size goes to zero: $g(a) \to \infty$ if $a \to 0$. At the point of intersection in Fig. 3, the density of dangerous flaws gets so high that the volume of the specimens is completely filled with flaws and, left to that point, the "volume of dangerous flaws" even exceeds the volume of the specimens. For obvious reasons this is not possible in real materials[#7] and it is expected that in this case and for materials with $m > 6$, the relative frequency is (much) lower than the one implicitly predicted by Weibull theory.

Another inconsistency is caused by the fact, that the derivation of the fracture statistics (Eqs. (1) and (2)) assumes non-interacting flaws. This will only be true in the case of a low flaw density. If fracture statistics is applied for very small specimens made of a Weibull material the density of dangerous flaws gets high and the interaction between flaws cannot be neglected any longer. For that case some theoretical work indicates a loss of the size effect,[39] but an experimental proof is missing up till now.

In summary, it can be concluded that a Weibull material cannot exist in reality because for very small flaws the flaw density in a Weibull material gets so high that some interaction among the flaws must occur. But an interaction among the flaws is in conflict with the weakest link hypothesis, which is the fundamental assumption in the statisistical theory of brittle fracture. Interaction between flaws might cause some interlinking and might reduce the strength.[39] This fact is only relevant for very small specimens and components (having an effective volume of 10^{-4} mm^3 or less) and it is irrelevant for normal specimens. Therefore, it has not been discussed in the literature yet. But for components of the microsystem technique or for microelectronic devices this behaviour becomes highly relevant.

4. Conclusions

- For a long time it has been well known that there is a strong correlation between the strength and flaw distribution. One of the consequences of the statistical nature of strength in brittle materials is the size effect on strength. Its existence has been predicted theoretically[5,16] and has also been demonstrated in many experimental studies, e.g. in [1,4,19,33]. But a detailed fractographic analysis of flaw populations, especially of the relative size frequency of dangerous flaws, is missing. This is an important area for future work.
- The theoretical analysis made in this paper, made on a set of published experimental data, shows a very strong dependency of the flaw density on the flaw size ($g(a) \sim a^{-8}$). The observed type of flaw distribution (inverse power law) is a necessary condition for the occurrence of a Weibull distribution. But such a strong dependency of the flaw density on the flaw size can only be valid in a relatively small interval of flaw sizes. For example, for the data mentioned above, a change (scatter) in strength of a factor of about 3 implies a change in the size of the critical flaw sizes of a factor of about 10 and a change in the density of flaws of about 7 orders of magnitude.
- In particular, the slope of the flaw size density attributed to Weibull materials cannot be true for very small flaws, where the inverse power law predicts unrealistic high flaw densities. Therefore, the Weibull theory should fail in the case of very small flaws being fracture origins.
- The high flaw density of very small flaws attributed to Weibull materials should result in a strong interaction among the flaws. This is in discrepancy to the weakest link hypothesis, which is the fundamental assumption in the statistical theory of brittle fracture[4,5,8,12] and in particular, in the Weibull theory.[4,5] Therefore, the Weibull theory should fail if very small flaws are fracture origins.
- The situation discussed in the last two paragraphs will occur if very small specimens are tested. For modern ceramic materials having a Weibull modulus between $m = 10$–20 this might be relevant for specimens having an effective volume of about $V \approx 10^{-4}$ mm^3 or smaller.

[#7] For materials with $m < 6$ (with a very large scatter of strength) an analogue inconsistency occurs for very large specimens. Then – for a Weibull material – the size of the critical flaw should become larger than the length of the specimen. Of course this is not possible and indicates another limitation of the Weibull theory.

- For very small specimens (components) the flaw density should be lower than predicted by the Weibull theory and some interaction among the flaws should occur. This should cause a lower strength than predicted by the Weibull theory.
- For the design of components of the microsystem technique or of microelectronic devices a new statistical theory of brittle fracture has to be developed.
- Although all arguments were made for materials containing volume flaws, similar results can be expected for specimens or components failing from surface or edge flaws.

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References

1. Munz, D. and Fett, T., *Ceramics: Mechanical Properties, Failure Behaviour, Materials Selection*. Springer Verlag, Berlin/Heidelberg/New York, 1999.
2. Wachtmann, J. B., *Mechanical Properties of Ceramics*. John Wiley & Sons, New York, 1996.
3. Danzer, R., Mechanical performance and lifetime prediction. In *Concise Encyclopedia of Advanced Ceramic Materials*, ed. R. J. Brook *et al.* Pergamon Press, Oxford, 1991, pp. 286–299.
4. Danzer, R., Börger, A., Villanueva, M. A. R. and Supancic, P., Ein einfacher Festig-keitsversuch für Scheiben aus sproden Werkstoffen. *Materialwissenschaft Werkstofftechnik*, 2003, **34**, 490–498.
5. Weibull, W., *A Statistical Theory of Strength of Materials*. Royal Swedish Institute for Engineering Research, Stockholm, 1939, 1–45.
6. Weibull, W., A statistical distribution function of wide applicability. *J. Appl. Mech.*, 1951, **18**, 293–298.
7. Gumbel, E., *Statistics of Extremes*. Columbia University Press, New York, 1958.
8. Freudenthal, A. M., Statistical approach to brittle fracture. In *Fracture, Vol II*, ed. H. Liebowitz. Academic Press, New York/London, 1968, pp. 591–619.
9. de Jayatilaka, A. S. and Trustrum, K., Statistical approach to brittle fracture. *J. Mat. Sci.*, 1977, **12**, 1426–1430.
10. Hunt, R. A. and McCartney, L. N., A new approach to fracture. *Int. J. Fract.*, 1979, **15**, 365–375.
11. McCartney, L. N., Extensions of a statistical approach to fracture. *Int. J. Fract.*, 1979, **15**, 477–487.
12. Danzer, R., A general strength distribution function for brittle materials. *J. Eur. Ceram. Soc.*, 1992, **10**, 461–472.
13. Danzer, R., Reisner, G. and Schubert, H., Der Einfluß von Gradienten in der Defekt-dichte und Festigkeit auf die Bruchstatistik von spröden Werkstoffen. *Z. Metall.*, 1992, **83**, 508–517.
14. Danzer, R., Lube, T. and Supancic, P., Monte Carlo simulations of strength distributions of brittle materials – type of distribution, specimen – and sample size. *Z. Metall.*, 2001, **92**, 773–783.
15. Sigl, L., Effects of the flaw distribution function on the failure probability of brittle materials. *Z. Metall.*, 1992, **83**, 518–523.
16. Kittl, P. and Diaz, G., Weibull's fracture statistics or probabilistic strength of materials: state of the art. *Res. Mech.*, 1988, **24**, 99–207.
17. Danzer, R., Mechanical failure of advanced ceramics: the value of fractography. *Key Eng. Mat.*, 2002, **223**, 1–18.
18. Hampel, H. and Wiest, W., Structural analysis and life prediction for gas turbine components for the Mercedes Benz research car 2000. ASME, 1986, Paper No. 86-GT-199.
19. Soma, T., Ishida, Y., Matsui, M. and Oda, I., Ceramic component design for assuring long-term durability. *Adv. Ceram. Mater.*, 1987, **2**, 809–812.
20. ENV 843-1, Advanced technical ceramics, monolithic ceramics, mechanical properties at room temperature. Part 1: Determination of flexural strength, 1995.
21. ENV 843-5, Advanced technical ceramics, monolithic ceramics, mechanical properties at room temperature. Part 5: Statistical analysis, 1996.
22. Danzer, R. and Lube, T., Fracture statistics of brittle materials: it does not always have to be Weibull statistics. In *Ceramic Materials and Components for Engine*, ed. K. Niihara *et al.* Japan Fine Ceramics Association, Tokyo, ISBN 4-9980630-0-6, 1998, pp. 683–688.
23. Vojta, A. and Clarke, D. R., Electrical impulse induced fracture of zinc oxide varistor ceramics. *J. Am. Ceram. Soc.*, 1997, **80**, 2086–2092.
24. Lengauer, M., Rubesa, D. and Danzer, R., Finite element modelling of the electrical impulse induced fracture of a high voltage varistor. *J. Eur. Ceram. Soc.*, 2000, **20**, 1017–1021.
25. Supancic, P., Mechanical stability of $BaTiO_3$-based PTC thermistor components: experimental investigations and theoretical modelling. *J. Eur. Ceram. Soc.*, 2000, **20**, 2009–2024.
26. Supancic, P. and Danzer, R., Mechanische Zuverlässigkeit von keramischen Kaltleiterbauelementen—Möglichkeiten zur Verbesserung. *Z. Metall.*, 2004, **95**, 607–617.
27. Balzer, B., Hagemeister, M., Kocher, P. and Gauckler, L., Mechanical strength and microstructure of zinc oxide varistor ceramics. *J. Am. Ceram. Soc.*, 2004, **87**, 1932–1938.
28. Dewitte, C., Elst, R. and Delannay, F., On the mechanism of delamination fracture of $BaTiO_3$-based PTC thermistor. *J. Eur. Ceram. Soc.*, 1994, **14**, 481–492.
29. Fulton, C. C. and Gao, H., Electromechanical fracture in piezoelectric ceramics. *Int. J. Fract.*, 1999, **98**, 17–22.
30. Börger, A., Supancic, P. and Danzer, R., The ball on three balls test for strength testing of brittle discs—stress distribution in the disc. *J. Eur. Ceram. Soc.*, 2002, **22**, 1425–1436.
31. Börger, A., Supancic, P. and Danzer, R., The ball on three balls test for strength testing of brittle discs. Part II: Analysis of possible errors in the strength determination. *J. Eur. Ceram. Soc.*, 2004, **24**, 2917–2928.
32. Lu, C., Danzer, R. and Fischer, F.D., Fracture statistics of brittle materials: Weibull or normal distribution. Phys. Rev. 2002, E 65, Article Number 067102, 1–4.
33. Lube, T., *Entwicklung eines Zugversuches zur Zuverlässigkeitsanalyse von keramischen Ventilen*. Dipl.-Ing. thesis, Montanuniversität Leoben, 1993.
34. Primas, R. J. and Gstrein, R., ESIS Round Robin on fracture toughness. *Fatigue Fract. Eng. Mater. Struct.*, 1997, **20**, 513–532.
35. Mecholsky Jr., J. J., Quantitative fractography: an assessment. *Ceram. Trans.*, 1991, **17**, 413–451.
36. Dusza, J. and Steen, M., Fractography and fracture mechanics property assessment of advanced structural ceramics. *Int. Mater. Rev.*, 1999, **44**, 165–216.
37. Morrell, R., Fractography of brittle materials, measurement good practice Guide No. 14, HMSO, National Physical Laboratory, UK, 1368-6550, 1999.
38. CEN/TS 843-6, Advanced technical ceramics, monolithic ceramics, mechanical properties at room temperature. Part 6: Guidance for farctographic investigation, 2004.
39. Lu, C., Danzer, R. and Fischer, F. D., Scaling of fracture strength in ZnO: effects of pore/grain-size interaction and porosity. *J. Eur. Ceram. Soc.*, 2004, **24**, 3643–3651.
40. Börger, A., *Eine Methode zur biaxialen Festigkeitsprüfung an Scheiben aus sprödem Werkstoff*. Thesis, Montanuniversität Leoben, 2004.